1. Double valued reflection

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involving a rectangular lattice. We give the Riemann map for the domain bounded by one branch of the associated real quartic (elliptic) curve in terms of a Weierstrass sigma quotient and the $\mathscr{P}$-function.

1. Double valued reflection

To place our work in context, we first consider, somewhat informally, the general concept of an anti-holomorphic involutive correspondence, or multiple valued reflection, on a complex manifold $\mathscr{U}$. This is an assignment $z \mapsto Q_z$, of a complex subvariety $Q_z \subset \mathscr{U}$ to each point $z \in \mathscr{U}$, such that

$$w \in Q_z \Leftrightarrow z \in Q_w . \tag{1.1}$$

The variety Q_z depends antiholomorphically on the point z in a way which can be made precise. The "fixed point set" is the set

$$\gamma = \{z \in \mathscr{U} \mid z \in Q_z\} . \tag{1.2}$$

Such a correspondence is double valued if each Q_z is generically zero dimensional and contains two points. Starting from a generic point $z_0 \in \mathscr{U}$, we have $Q_{z_0} = \{z_1, z_1'\}$. Choosing z_1, we get $Q_{z_1} = \{z_0, z_2\}$, $Q_{z_2} = \{z_1, z_3\}, \ldots$ Thus we generate a sequence

$$z_0 \mapsto z_1 \mapsto z_2 \mapsto z_3 \cdots , \tag{1.3}$$

with z_{2k} and z_{2k+1} locally determined and depending holomorphically, respectively, antiholomorphically, on z_0. If we choose z_1', then $Q_{z_1'} = \{z_0, z_2'\}$, $Q_{z_2'} = \{z_1', z_3'\}, \ldots$, and we generate a similar sequence

$$z_0 \mapsto z_1' \mapsto z_2' \mapsto z_3' \cdots . \tag{1.4}$$

A basic problem of the theory is to understand the dynamics of this process. We shall make the foregoing more precise, but only in the case where $\mathscr{U}$ is an open subset of the complex plane.

Let $r(z, \zeta)$ be holomorphic on $\mathscr{U} \times \bar{\mathscr{U}}$ where

$$\mathscr{U} \subseteq \mathbf{C}, \quad \bar{\mathscr{U}} = \{\bar{z} \mid z \in \mathscr{U}\} , \tag{1.5}$$

which satisfies

$$r \circ \rho = \bar{r}, \quad \rho(z, \zeta) = (\bar{\zeta}, \bar{z}) . \tag{1.6}$$

We set

$$\Gamma = \{(z, \zeta) \in \mathcal{U} \times \bar{\mathcal{U}} \mid r(z, \zeta) = 0\}, \tag{1.7}$$

$$\gamma = \{z \in \mathcal{U} \mid r(z, \bar{z}) = 0\}, \tag{1.8}$$

$$Q_w = \{z \in \mathcal{U} \mid r(z, \bar{w}) = 0\}. \tag{1.9}$$

By (1.6) $r(z, \bar{z})$ is real on $\mathcal{U}$, and γ is a real analytic curve. Γ, the complexification of γ, is invariant under the anti-holomorphic involution ρ, which has $\Gamma \cap \{\zeta = \bar{z}\} \cong \gamma$ as fixed-point set. We denote the projections, restricted to Γ, by

$$\pi_1(z, \zeta) = z, \quad \pi_2(z, \zeta) = \zeta, \quad \pi_2 = \overline{\pi_1 \circ \rho}. \tag{1.10}$$

The multiple valued reflection $z \to Q_z$ on $\mathcal{U}$ is derived from the single valued reflection ρ on Γ by

$$z \mapsto \pi_1^{-1}(z) \to \rho(\pi_1^{-1}(z)) \to \pi_1(\rho(\pi_1^{-1}(z))) \equiv Q_z.$$

If $z_0 \in \gamma$ and $r_z(z_0, \bar{z}_0) \neq 0$, then the holomorphic implicit function theorem gives a unique w near z_0, depending anti-holomorphically on z near z_0, and satisfying $r(z, \bar{w}) = 0$. The map $z \mapsto w$ is the local reflection in γ in the form emphasized by Caratheodory [1].We are merely considering this from a more global point of view.

DEFINITION. The real curve γ admits double valued reflection, if the two holomorphic maps $\pi_1 : \Gamma \mapsto \mathcal{U}$, $\pi_2 : \Gamma \to \bar{\mathcal{U}}$ are twofold branched coverings. (In case Γ has singularities, we may replace it by its Riemann surface in this definition.)

The maps which interchange the fibers of the maps π_i are denoted by $\tau_i : \Gamma \mapsto \Gamma$, $i = 1, 2$,

$$\pi_i \circ \tau_i = \pi_i, \ (\tau_i)^2 = id, \ i = 1, 2; \ \tau_2 \circ \rho = \rho \circ \tau_1. \tag{1.11}$$

They are holomorphic maps which don't commute in general. Their commutator is σ^2,

$$\sigma = \tau_1 \circ \tau_2. \tag{1.12}$$

The map σ is *reversible,* i.e. conjugate to its inverse via an involution: $\sigma^{-1} = \tau_2 \tau_1 = \tau_1 \sigma \tau_1$.

We can now explain the sequences (1.3) and (1.4) more precisely in terms of σ. From

$$r(z_0, \bar{z}_1) = r(z_2, \bar{z}_1) = r(z_2, \bar{z}_3) = 0,$$

we have $\tau_2(z_0, \bar{z}_1) = (z_2, \bar{z}_1)$, and

$$\sigma(z_0, \bar{z}_1) = \tau_1(z_2, \bar{z}_1) = (z_2, \bar{z}_3) .$$

Hence, the map $z_0 \mapsto z_2$ is the first component of $(z_0, \bar{z}_1) \mapsto \sigma(z_0, \bar{z}_1)$. Thus, we are led to studying the iterates σ^k of a reversible holomorphic map σ on a Riemann surface Γ.

In this paper we shall concentrate on the algebraic case. Thus, we assume $\mathscr{U} = \mathbf{C}$, and take $r(z, \zeta)$ to be a holomorphic polynomial of two complex variables. For a double valued reflection we must have

$$\deg_z r = \deg_\zeta r = 2, \deg r \leqslant 4 . \tag{1.13}$$

The condition is thus very restrictive. In fact, one can give give a complete classification. We write

$$r(z, \bar{z}) = b_0 + b_1 z\bar{z} + b_2 z^2 \bar{z}^2 + 2Re(a_0 z + a_1 z^2 + a_2 z^2 \bar{z}) , \tag{1.14}$$

where the b's are real, the a's complex, constants. The form of r is invariant under linear transformation $z \mapsto cz + d$. The form of the equation $r = 0$ is also invariant under inversion

$$z \mapsto z^{-1}, r(z, \bar{z}) \mapsto (z\bar{z})^2 r(z^{-1}, \bar{z}^{-1}) , \tag{1.15}$$

and hence, under all Moebius transformations. (1.15) results in the change of coefficients

$$(b_0, b_1, b_2; a_0, a_1, a_2) \mapsto (b_2, b_1, b_0; \bar{a}_2, \bar{a}_1, \bar{a}_0) . \tag{1.16}$$

The family of curves $r = 0$ includes the conics, the lemniscates (inversions of hyperbolas), cuspidal cubics (inversions of parabolas), as well as certain elliptic curves.

If we assume that γ is non-empty, then a translation results in $b_0 = 0$. An inversion then results in $b_2 = 0$. We assume that $\deg r = 3$, so that $a_2 \neq 0$ (otherwise, we have a conic, which case we shall treat in the next section). If both $b_0 = 0$ and $a_0 = 0$, we can still invert and reduce γ to a conic. Thus, we assume that either $b_0 \neq 0$, or $a_0 \neq 0$. The translation $z \mapsto z + c$ results in

$$a_1 \mapsto a_1 + \bar{c} a_2 ,$$

so that we can make $a_1 = 0$ by a unique choice of c. Now we can make the changes

$$z \mapsto cz, r \mapsto \lambda r, \lambda = \bar{\lambda} \neq 0, c \neq 0 ,$$

which result in

$$a_2 \mapsto \lambda c^2 \bar{c} a_2 .$$

We make $a_2 = 1$ and then restrict to $c = \bar{c}$, $\lambda c^3 = 1$. If $b_0 \neq 0$, we can make $b_0 = 1$, which gives the normal form, under the Moebius group,

$$r = 1 + bz\bar{z} + 2Re(az + z^2\bar{z}),\ b \in \mathbf{R},\ a \in \mathbf{C}\ . \tag{1.17}$$

If $b_0 = 0$, we have

$$a_0 \mapsto \lambda c a_0 = c^{-2} a_0\ .$$

We make $|a_0| = 1$, after which we must restrict to $c = \pm 1$, $\lambda = \mp 1$. The normal form is

$$r = bz\bar{z} + 2Re(az + z^2\bar{z}),\ |a| = 1,\ b \geqslant 0\ . \tag{1.18}$$

In summary we have proved the following.

PROPOSITION 1.1. *Suppose that the (non-empty) real algebraic curve $\gamma \subset \mathbf{C}$ admits double valued reflection. Then, under Moebius transformation γ is equivalent to a conic section, or to a curve $r = 0$, where r is given by either (1.17) or (1.18).*

A different normal form will appear later from the intrinsic point of view.

We note that if $\deg_z r = 1$, then we have a circle, and the above process reduces to transforming it to a straight line. These are the cases of global single valued reflection.

2. CONIC SECTIONS

In this section we shall describe the relevant geometry of real quadratic curves in the complex plane. This should give a clearer idea of the possible dynamics in (1.3) and (1.4). The description is only "local" in that it depends on making certain branch cuts, so that the double valued reflection falls into two single valued reflections. In the next section we shall give a more coherent treatment, essentially by passing to a two-sheeted Riemann surface, namely Γ, on which the double valued reflection becomes a single valued one, namely ρ.

The conic with foci $\pm a$, $a > 0$, and parameter $b > 0$ is given by

$$|z + a| + \varepsilon |z - a| = b,\ \varepsilon^2 = 1\ . \tag{2.1}$$

This is an ellipse if $\varepsilon = +1$, $b > 2a$, and one branch of a hyperbola if $\varepsilon = -1$, $b < 2a$. The other branch is gotten by replacing b with $-b$. Squaring and simplifying twice gives the equation