5. The general case

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

Therefore there is a commuting diagram

$$\begin{array}{ccc} \Gamma & \subset & \tilde{\Gamma} \\ \cup & & \uparrow \\ G & \to & \hat{G} \end{array}$$

Where $\hat{G} = \frac{G * \langle s \rangle}{\langle\langle w \rangle\rangle}$. Thus $G \to \hat{G}$ is injective. □

REMARK. The alert reader will have noticed that the hypotheses of theorem 4.4 can be weakened. All that has been used is that the coefficients of w are of infinite order. Indeed a careful examination of the proof yields the following sharper statement. If $G \to \frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$ is not injective then one of the *separating* coefficients in w has finite order (separating means between a t and a t^{-1}).

5. THE GENERAL CASE

In this section we consider the adjunction problem as stated in the introduction, in its full generality. We continue to work with torsion-free groups. We shall introduce a class of words with exponent not necessarily 1 for which the methods of the previous sections can be adapted to provide a solution to the adjunction problem. We call such words *amenable*. Before defining amenability in general we shall consider a class of simpler words, on which the general definition will be based, these we call *suitable* words.

t-SHAPES, t-SEQUENCES AND SUITABILITY

Consider finite sequences whose elements are t or t^{-1}. We call such a sequence a *t-sequence.* If m is a positive integer let t^m denote the sequence $t, t, \ldots, t$, m times and let t^{-m} denote the sequence $t^{-1}, t^{-1}, \ldots, t^{-1}$, m times. A *clump* is a maximal connected subsequence of the form t^m or t^{-m} where $m > 1$ and these are said to have *order m* and $-m$ respectively. We call a clump of positive order an *up* clump and a clump of negative order a *down* clump. A sequence is *suitable* if it has exactly one up clump which is not the whole sequence and possibly some down clumps, or if it has exactly one down clump which is not the whole sequence and possibly some up clumps.

It follows that, after a possible cyclic rotation or change $t \mapsto t^{-1}$, a suitable sequence has the form

$$t^s t^{-r_0} t t^{-r_1} t \ldots t t^{-r_k}$$

where $s > 1$, $k \geqslant 0$ and $r_i \geqslant 1$ for $i = 0, \ldots, k$. Notice that the power sequences t^n, $n > 1$ and the alternating sequences $tt^{-1}tt^{-1} \ldots tt^{-1}$ are not suitable.

Any word in $G * \langle t \rangle$ has a t-sequence associated to it, given by its t-shape. Since we are only interested in words up to cyclic permutation we shall say that a word is *suitable* if after a cyclic permutation its associated sequence is suitable.

THEOREM 5.1. *Suppose that* G *is torsion-free and that* $w \in G * \langle t \rangle$ *is a suitable word. Then* w *has a solution over* G.

Proof. We shall prove the theorem directly without using the algebraic trick used in the last section. Suppose that G does not inject in $\frac{G * \langle t \rangle}{\langle\langle w \rangle\rangle}$. By the second transversality lemma there is a cell subdivision of the 2-sphere such that, the 1-cells of K are oriented, the 2-cells are of the two types I and I' illustrated in figure 7 with the corners labelled by elements of G and such that the clockwise product of the corner labelling around any 0-cell is 1 except for one vertex (where it is non-trivial). Assume that K is minimal with these properties.

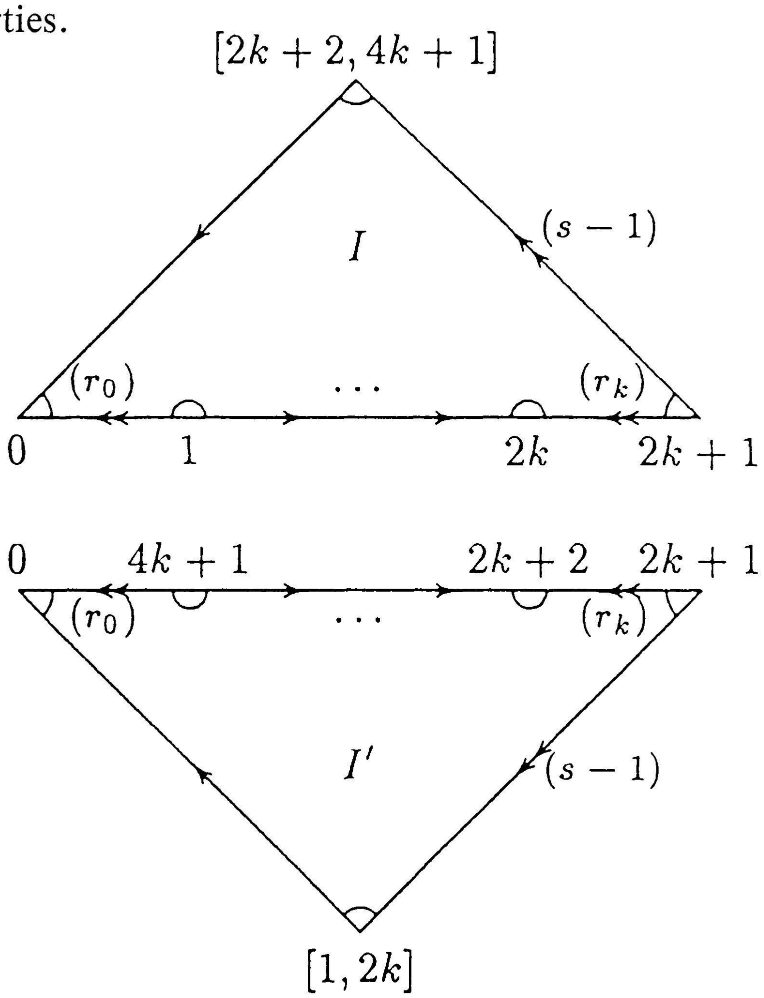

FIGURE 7
The 2-cells I, I'

In figure 7 we have used double arrows labelled by (r) as an abbreviation for r consecutive single arrows (corresponding to t^r). For example the top right double arrow in cell I represents $s-1$ single arrows.

We shall organise a traffic flow on K as in the proof of theorem 4.1 and having precisely the same four properties that were listed there. Then there can only be complete crashes at maxima or minima as before and such a crash at a vertex with trivial labelling implies a simplification of the diagram, just as before.

The traffic flow is defined in a very similar way to the flow in theorem 4.1. The unit times for the motion are given in figure 7. We have to explain what happens inside the clumps. The idea is to treat each of these as a uphill or downhill section along which a car travels in unit time pausing very briefly at each internal (stopping) vertex. The exception is the one up clump in cells of type I, where the car parks at the last possible stopping vertex while most of the motion takes place (in the other half of the cells of type I'), with a similar (reversed) motion in cells of type I'. □

We now combine the theorem with the algebraic trick of the last section.

NORMAL FORM AND AMENABILITY

There is a definition of a *normal form* for a word based on any t-sequence, which is similar to the form of lemma 4.2 (which can be regarded as defining the normal form for the t-sequence $t^{-1}tt^{-1}t\ldots t^{-1}t^2$).

To be precise, the normal form associated to a t-sequence is obtained as follows: arrange the t's and t^{-1}'s in the given sequence in anticlockwise order around a circle with a vertex between each pair. Put arrows anticlockwise next to each t and clockwise next to each t^{-1}. The vertices are now of the four types (top, bottom, up, down) defined in the proof of theorem 4.1. Write a letter c next to each up and each down vertex, a letter b next to each top vertex and a letter a next to each bottom vertex. Reading the letters round the circle anticlockwise (starting anywhere) gives the required normal form, where the letters a, b, c are interpreted as generic elements of Y, X, J respectively, where X, Y, J have the same meanings as in lemma 4.2.

A word is *amenable* if it can be conjugated to a word in normal form for a *suitable* t-sequence (as defined above the theorem). Notice that amenability is again a property of the t-shape of a given word and that lemma 4.2 proves that all words of exponent ± 1 are amenable.

THEOREM 5.2 (General case of theorem 1.1). *Suppose that* G *is torsion-free and that* $w \in G * \langle t \rangle$ *is an amenable word. Then* w *has a solution over* G.

Proof. The proof is a straightforward combination of the proof of theorem 5.1 with the proofs given in the last section. This completes the proof of theorem 1.1 announced in the introduction. □

REMARK. The reader can check that the proof of theorem 5.1 can be adapted to a more general class of *suitable* words, namely words with several up and down clumps, which are not interleaved. However it can be proved that the corresponding notion of *amenable* words is exactly the same as that given above so there is no point in pursuing this.

REMARKS ON AMENABILITY

We believe that Klyachko's methods can, with further extension, be adapted to give a solution to the adjunction problem for torsion-free groups in general and we intend to pursue this in a later paper on the subject. However, as we have seen, his methods extend without too much work to the case of amenable words and we finish this section with a brief discussion of amenability, and in particular, consider how general is this class of amenable words.

In some sense (see below) nearly all t-sequences are amenable. However it is definitely *not* the case that *all* sequences are amenable. We now give some examples.

There are 17 t-sequences of lengths $\geqslant 2$ and $\leqslant 8$ (up to cyclic permutation, inversion and replacement of t by t^{-1}) which fail to be amenable while 30 are amenable. Examples of amenable t-sequences are $t^3t^{-1}tt^{-1}tt^{-1}$, $t^4t^{-2}tt^{-1}$, $t^2t^{-1}tt^{-2}tt^{-1}$ and examples of non-amenable sequences are t^8, $t^2t^{-2}t^2t^{-2}$, $t^3t^{-1}t^3t^{-1}$. Note that the 17 non-amenable sequences include several sequences for which the adjunction problem is solved, see below.

As length increases, the situation becomes progressively better and it can be checked that the proportion of non-amenable sequences tends to zero as length tends to ∞. Up to and including length 9, there is no difference between suitable and amenable sequences, but as length increases the difference becomes immense, with again very few sequences suitable compared with (nearly all) amenable. The first example of a t-sequence which is amenable but not suitable is $t^2t^{-1}t^2t^{-2}tt^{-2}$ and a more typical (longer) example is

$$t^3t^{-1}t^3t^{-2}tt^{-2}t^5t^{-4}.$$

The t-sequence t^n is interesting because the adjunction problem is already proved (without the torsion-free hypothesis) for words with this t-shape [L]. However the methods discussed here do not extend this result to t-sequences in normal form based on t^n. An example is $t^3t^{-1}t^3t^{-1}$.

Another interesting case is the sequence tt^{-1} which is not amenable. However a simple trick (substitute u^2 for t) makes it suitable. Hence theorem 5.1 implies a solution to the adjunction problem (over torsion-free groups) for words of the form $gtg't^{-1}$. For words of this shape, torsion-free is a necessary condition as the example in the introduction shows!

We do not yet have a simple test for amenability though it is easy from the definition to write down large classes of amenable sequences. However it can be seen that, speaking very roughly, a sequence is amenable unless it is has a uniform slope, like $t^5t^{-3}t^5t^{-3}$ or $t^3t^{-3}t^3t^{-3}$ (slope zero).

6. Further Applications

We give here the other applications from [Kl] of the crash theorems, not covered above.

Theorem 6.1 (Application to free products). *Let A, B be groups and suppose each (cyclic) factor of $u \in A * B - A$ has infinite order. Then the natural homomorphism $A \to \langle A * B \mid [A, u] = 1 \rangle$ is injective.*

Proof. Suppose not. Then the conditions of the first transversality lemma apply and there is a non trivial element $a \in A$ such that $a \in \langle\langle [A, u] \rangle\rangle$. So we have a cell subdivision K of the 2-sphere such that reading round from the base point $*$ for every 2-cell in K spells out the word

$$w(a) = (c_0^{-1}ac_0)c_1 \cdots c_{n-1}(c_n^{-1}a^{-1}c_n)c_{n-1}^{-1} \cdots c_1^{-1}$$

for some $a \in A$, see figure 8. Note that if this 2-cell has the opposite orientation then the word spelt out is $w(a^{-1})$.

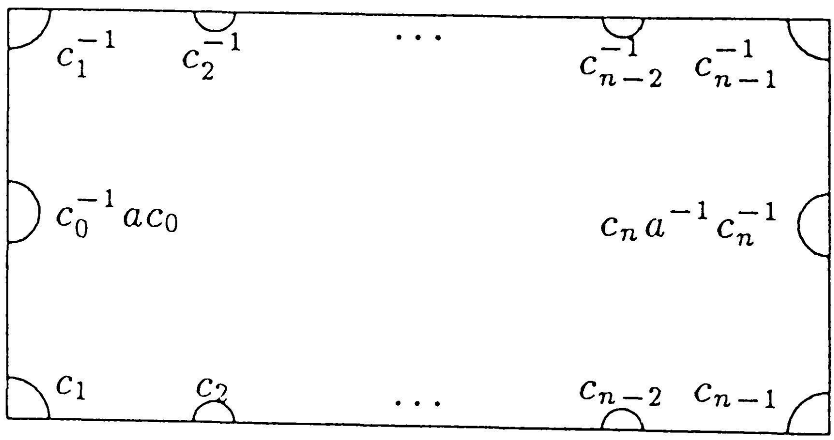

Figure 8
The 2-cell labelled by $w(a)$