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CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID

by Roger FENN, Dale ROLFSEN and Jun ZHU [1])

ABSTRACT. The centre of the braid group B_n is well-known to be infinite cyclic and generated by a twist braid. In this paper we consider the centraliser of certain important subgroups, and in particular we characterise the elements of B_n which commute with one of the usual generators σ_j. This characterisation is generalised to the monoid of singular braids SB_n, recently introduced (independently) by J. Baez and J. Birman. We determine the singular braids which commute with σ_j, or with a singular generator τ_j; in fact we show these submonoids are the same.

We establish that the centraliser in B_n of σ_j is isomorphic to the cartesian product of two groups: the group of $(n-1)$-braids whose permutations stabilise j and the group of integers. More generally, we show that the centraliser of the naturally-included braid subgroup $B_r \subset B_n$ likewise splits as a direct product, and we give an explicit presentation for this centraliser. We also describe the centralisers of $SB_r \subset SB_n$.

As another application we consider a conjecture of J. Birman regarding the injectivity of a map, related to Vassiliev theory, $\eta : SB_n \to \mathbf{Z}B_n$ from the singular braid monoid to the group ring of the braid group. We see that the question is related to the centraliser problem and prove the injectivity of η for braids with up to two singularities.

1. INTRODUCTION AND BASIC DEFINITIONS

The braid group B_n, for an integer $n \geqslant 2$, may be considered abstractly as the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_j \sigma_k = \sigma_k \sigma_j \qquad \text{if} \quad |j-k| > 1,$$
$$\sigma_j \sigma_k \sigma_j = \sigma_k \sigma_j \sigma_k \qquad \text{if} \quad |j-k| = 1 .$$

There are equivalent geometric descriptions of braids as strings in space, as automorphisms of a free group F_n, as the fundamental group of a configuration space, or as homeomorphisms of an n-punctured plane (see below), which explains the importance of the braid groups in many

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disciplines. The originator of braid theory, Emil Artin, posed several (at the time) "unsolved problems" in [Art2], including:

"With what braids is a given braid commutative?"

"Decide for any two given braids whether they can be transformed into each other by an inner automorphism of the group."

The present paper is concerned primarily with the first question: finding the centraliser of a given braid. Although an algorithm exists, as we'll describe shortly, this problem can still be said to be open in general. However, we consider the most basic special case and characterise, in a simple geometric way, the set of all braids which commute with one of the generators σ_j.

The latter question — the conjugacy problem — was settled in principle by Garside [Gar], who gave a finite procedure to decide if two given braids are conjugate. The present work also contributes to this question, in that we determine exactly which inner automorphisms will take one of the standard generators to another.

G. Burde [Bur] has computed the centralisers of certain special kinds of braids: those which are "j-pure" as defined by Artin, meaning a pure braid (see our discussion below) for which all strings except the j^{th} are horizontal straight lines. Burde's point of view (like ours, which was developed independently) is partly algebraic and partly geometric.

For an arbitrarily given element $\alpha \in B_n$, there is an algorithm to find a finite set of generators for its centraliser, as shown by G. Makanin [Mak]. This result was extended by G. Gurzo [Gur1] to the centraliser of any finite set of elements of B_n, who showed that the generators can be taken to be positive braids (no negative exponents). The methods of Makanin and Gurzo are algebraic and combinatorial. They rely heavily on techniques pioneered by Garside; transferring the problem to the monoid of positive braids, and thus making its solution a finite search. Their method sheds little light on the actual structure of the centraliser. However, in a later paper [Gur2], Gurzo extended the work to explicitly compute generating sets for centralisers of various special types of braids, including the σ_j and their powers. As an application, she discovered that the centraliser of any nonzero power σ_j^m is independent of m.

In fact more can be said, in general, of centralisers of finite sets in B_n; they are *biautomatic*. Thurston proved in [ECHLPT] that B_n is biautomatic (see also [Char1], [Char2]), and Gersten-Short [GS] have shown that centralisers of finite sets in biautomatic groups are themselves biautomatic.

Following some preliminaries, our goal in Sections 2 and 3 will be to characterise the centraliser of σ_j using the geometric viewpoint, exploiting the action of B_n on (classes of) arcs in the complex plane. More generally, we identify all solutions β to $\sigma_j \beta = \beta \sigma_k$ by a natural criterion involving braids as geometrical objects — having what we call a "(j, k)-band." Using this criterion, we recover Gurzo's result that the centraliser of σ_j^m is independent of $m \neq 0$. It also gives an alternative proof to the old result [Chow] that the centre of B_n is infinite cyclic. In Section 4 we use our results to describe the structure of the centraliser of B_r in B_n, where $B_r \subset B_n$ is the usual inclusion, $r \leqslant n$, and we give an explicit presentation of this centraliser (our generators are different from Gurzo's).

In Section 5 we consider an extension of B_n to the singular braid monoid SB_n recently introduced by Birman [Bir2] and Baez [Bae] to study Vassiliev theory. We show that the centraliser of a basic singular generator τ_j in SB_n coincides with the centraliser of σ_j. Moreover, the solutions β to $\tau_j \beta = \beta \tau_k$ are shown to be exactly those β which have a (possibly singular) (j, k)-band.

Our results are used in Section 6 to study a question raised by Birman regarding injectivity of the Baez-Birman-Vassiliev map η from SB_n into the group ring $\mathbf{Z}B_n$. Finally, in Section 7 we generalise the "Band Theorem" (2.2) to the context of singular braids, and consider the centralisers of σ_j, τ_j and SB_r in SB_n.

We would like to thank Joan Birman, Vaughan Jones, Christine Riedtman and Hamish Short for helpful conversations regarding this work.

GEOMETRIC BRAIDS. Let $\mathbf{C}$ denote the complex plane, $\{1, \ldots, n\}$ the first n integral points on the positive real axis and $\mathbf{I} = [0, 1]$ the unit interval. We consider an n-braid β to be a collection of n disjoint strings $\beta \subset \mathbf{C} \times \mathbf{I} = \{(z, t)\}$ such that the j-th string runs, monotonically in t, from the point $(j, 0)$ to some point $(k, 1), j, k \in \{1, \ldots, n\}$. An isotopy in this context is a deformation through braids (with fixed ends), and isotopic braids are considered equivalent. We write $j = \beta * k$ or, equivalently $j * \beta = k$, so that braids can act either on the right or left as permutations of $\{1, \ldots, n\}$. A *pure* braid is one whose permutation is the identity. We will picture braids horizontally rather than vertically, so that multiplication of braids is by concatenation from left to right, just as written algebraically. The (equivalence classes of) braids with this multiplication form the group B_n described algebraically above.

Basic references for braid theory are [Art1], [Art2] and [Bir1]; [BZ] and [Han] also contain good accounts and [Bir2] is an up-to-date discussion including singular braid theory. As noted by Artin, one can also regard a braid as corresponding to a homeomorphism of **C** onto itself, with compact support and setwise preserving $\{1, ..., n\}$. More precisely, a braid corresponds to a *mapping class* in which homeomorphisms of **C** are considered equivalent if one can be obtained from the other by a (compactly supported) isotopy in which the points $\{1, ..., n\}$ are held fixed. Thus equivalence classes of braids are in one-to-one correspondance with such mapping classes. As depicted in Figure 1 (see also Figure 6), the braid σ_j corresponds to the class of a homeomorphism which is a local (right-hand) twist of the plane interchanging the points j and $j+1$, and supported on a neighbourhood of the interval $[j, j+1]$.

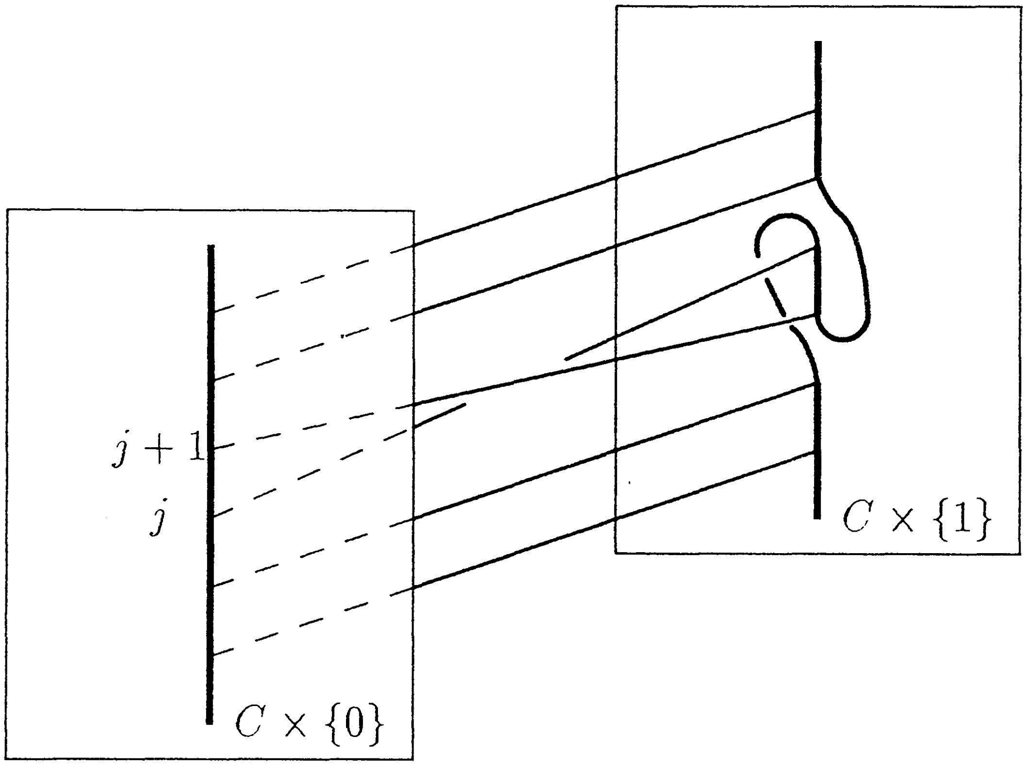


FIGURE 1
The homeomorphism associated with a generator σ_j

The inverse correspondence is as follows: suppose one has a homeomorphism of **C** which is compactly supported and fixes $\{1, ..., n\}$ setwise. This homeomorphism is isotopic to the identity, but the points $\{1, ..., n\}$ may move during the isotopy. The track of these points, in $\mathbf{C} \times \mathbf{I}$, through the isotopy, gives the geometric braid corresponding to (the class of) the given homeomorphism.

The product of braids corresponds to composition of homeomorphisms of **C**. One can have the braid group act on **C** either on the left or right — both conventions appear in the literature. It is convenient for us, in fact, to adopt both conventions, extending the above notation for permutations so that

$$*\beta : \mathbf{C} \to \mathbf{C} \text{ corresponds to a mapping } \mathbf{C} \times 0 \to \mathbf{C} \times 1 ,$$

and defines an action on the right, whereas

$$\beta * : \mathbf{C} \to \mathbf{C} \text{ corresponds to a mapping } \mathbf{C} \times 1 \to \mathbf{C} \times 0$$

and operates on the left. Thus, for any subset X of $\mathbf{C}$, and braids α, β:

$$X * (\alpha\beta) = (X * \alpha) * \beta,$$
$$(\alpha\beta) * X = \alpha * (\beta * X), .$$
$$X * \beta = \beta^{-1} * X$$

This action extends the permutation action of B_n as discussed earlier. (We note that our depiction of generators σ_j disagrees with that of some earlier authors, but is in keeping with recent practice, so that σ_j corresponds to a "positive" oriented crossing; a right-handed twist instead of left.)

PROPER ARCS AND RIBBONS. An important rôle will be played by the set of arcs in $\mathbf{C}$ which are *proper rel* $\{1, \ldots, n\}$, by which we mean that their endpoints are in the set $\{1, \ldots, n\}$ and their interiors are disjoint from that set. Such an arc from (say) j to k is called a (j, k)-arc. We consider two (j, k)-arcs equivalent if they are connected by a continuous family of proper arcs; in other words, isotopic. Unless otherwise stated, we do *not* distinguish a (j, k)-arc A from its *reverse,* the oppositely oriented $\bar{A}$, which is a (k, j)-arc. Use the notation:

$\mathbf{A}_n$ = the set of proper arcs in $\mathbf{C}$, modulo isotopy fixing $\{1, \ldots, n\}$.

It is clear from the above discussion that the braid group also acts naturally on $\mathbf{A}_n$, and we adopt the same symbols $\beta *$ and $* \beta$ for the left and right actions.

By a *ribbon* we will mean an embedding

$$R : \mathbf{I} \times \mathbf{I} \to \mathbf{C} \times \mathbf{I}$$

such that $R(s, t) \in \mathbf{C} \times t$. Suppose one has a braid β and a (j, k)-arc A in $\mathbf{C} \times 0$. Then the isotopy corresponding to β moves A through a ribbon which is *proper* for β, meaning $R(0, t)$ and $R(1, t)$ trace out two strands of the braid, while the rest of the ribbon is disjoint from β. The left end of the ribbon is A and the right end is $A * \beta$.

1.1 PROPOSITION. *Let* β *be an* n*-braid and* A *and* B *be proper arcs for* $\{1, \ldots, n\}$. *Then* $A * \beta = B$ *if and only if there is a proper ribbon for* β *connecting* $A \subset \mathbf{C} \times 0$ *to* $B \subset \mathbf{C} \times 1$.

Proof. We have already argued that $A * \beta = B$ implies the existence of the ribbon. On the other hand, suppose there is a ribbon R from A to B proper for β. Then by reflection of the ribbon from A to $A * \beta$ and concatenation with R, one has a ribbon from $A * \beta$ to B along $\beta^{-1}\beta$. But $\beta^{-1}\beta$ can be moved by level-preserving isotopy to the trivial braid $\{1, \ldots, n\} \times \mathbf{I}$, and then the image of the ribbon provides an isotopy from $A * \beta$ to B fixing $\{1, \ldots, n\}$. $\square$

2. Commutation and stabilisers

The theme of this paper is to reflect algebraic properties of a braid in the geometry of ribbons and the action of B_n on $\mathbf{A}_n$.

Consider an n-braid β which is constructed from an $(n-1)$-braid by running a narrow ribbon along the j^{th} string, with the ends of the ribbon being straight line segments on the real line, as pictured in Figure 2. The ribbon may be twisted arbitrarily. Let β consist of the two edges of the ribbon, together with the other strands of the $(n-1)$-braid (those of index greater than j need to be renumbered and have their ends shifted, of course.) Premultiplying β by σ_j corresponds to putting a twist in the left end of the ribbon, and the ribbon can be used to convey that twist through β until it emerges on the right, and we have the equation: $\sigma_j \beta = \beta \sigma_k$.

In the special case of $j = k$ we have constructed a class of braids which commutes with the generator σ_j. In fact, if β is any braid for which $[j, j+1] * \beta = [j, j+1]$, it can be isotoped, with fixed endpoints, into one with such parallel strands. Just slide the strands near each other along the ribbon, but taper to the identity to keep the ends fixed.

Definition. We say that β *has a (j, k)-band* if there exists a ribbon (the band) proper for β and connecting $[j, j+1] \times 0$ to $[k, k+1] \times 1$.

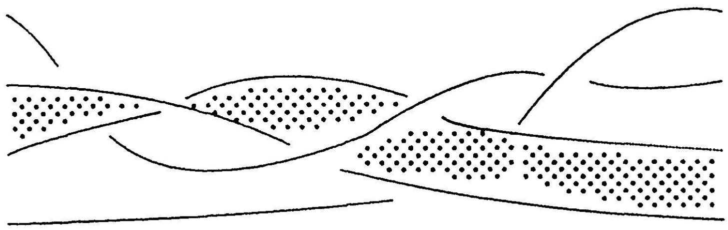


Figure 2
A braid with a (2.1)-band

According to Proposition 1.1, β has a (j,k)-band if and only if $[j,j+1]*\beta=[k,k+1]$. However, it may not be obvious, from an expression as a word in the generators, whether a braid has a (j,k)-band, and subwords of braids with bands may fail to have bands, as illustrated by the following example.

EXAMPLE. Consider the braids $\alpha=\sigma_2^{-1}\sigma_1^2\sigma_2$ and $\beta=\sigma_1\sigma_2^{-2}$. Then $\alpha\beta$ has a $(1,1)$-band. But neither α nor β have a $(1,1)$-band, although they both stabilise $\{1,2\}$. The arc $A=[1,2]*\alpha=\beta*[1,2]$ is as pictured in Figure 3. It is an interesting exercise for the reader to check that $\alpha\beta$ commutes with σ_1, whereas neither α nor β commutes with σ_1.

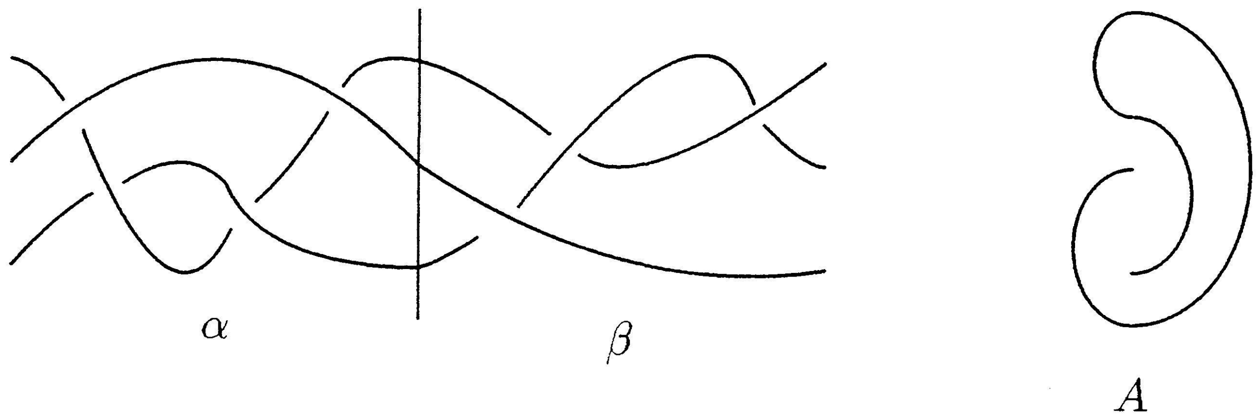


FIGURE 3

The braid $\alpha\beta=\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_1\sigma_2^{-2}$ and the arc $A=[1,2]*\alpha=\beta*[1,2]$

We can now formulate the central result of this paper.

2.1. THEOREM. *A braid* $\beta\in B_n$ *commutes with a generator* σ_j *if and only if it has a* (j,j)*-band. Equivalently, the action of* $*\beta$ *on* $\mathbf{A}_n$ *stabilises* $[j,j+1]$. □

This is an immediate corollary of a more general result.

2.2. THEOREM. *For a braid* $\beta\in B_n$ *the following are equivalent:*

(a) $\sigma_j\beta=\beta\sigma_k$,

(b) $\sigma_j^r\beta=\beta\sigma_k^r$, *for some nonzero integer* r,

(c) $\sigma_j^r\beta=\beta\sigma_k^r$, *for every integer* r,

(d) β *has a* (j,k)*-band,*

(e) $[j,j+1]*\beta=[k,k+1]$.

2.3. COROLLARY. *The centraliser of* σ_j^r *is independent of* $r\neq 0$ *and coincides with the stabiliser of the interval* $[j,j+1]$ *in the action of* B_n *upon* $\mathbf{A}_n$. □

2.4 COROLLARY. *The inner automorphism in* B_n *exchanging generators,* $\sigma_k = \beta^{-1}\sigma_j\beta$, *is achieved exactly by those braids* β *that have a* (j, k)*-band.* □

2.5 COROLLARY [Chow]. *The centre of* B_n, $n \geqslant 3$ *is infinite cyclic, generated by the braid* Δ^2, *where*

$$\Delta = \sigma_{n-1}(\sigma_{n-2}\sigma_{n-1}) \cdots (\sigma_1\sigma_2 \cdots \sigma_{n-1}) .$$

Proof. A braid commutes with all braid generators if and only if its action stabilises all the intervals $[1, 2], \ldots, [n-1, n]$, so it has a great ribbon containing the entire braid, connecting $[1, n] \times 0$ with $[1, n] \times 1$, necessarily in an order-preserving sense. Such a braid is clearly a multiple of the full-twist Δ^2. □

3. PROOF OF THEOREM 2.2

It is useful here to introduce an invariant of proper arcs. Throughout this section A will denote an oriented (k, l)-arc in $\mathbf{C}$ which is proper with respect to $\{1, \ldots, n\}$.

Associated with A is a word in the symbols $I_0, I_1, \ldots, I_n, I_0^{-1}, I_1^{-1}, \ldots, I_n^{-1}$ which can be described as follows. Assume that A is transverse to the real line. Starting from its initial point k, continue along A to l and whenever A crosses the interval $[m, m+1]$ write I_m if it crosses with increasing imaginary part and write I_m^{-1} otherwise. In the above notation, use the interval $(-\infty, 1]$ in case $m = 0$ and $[n, \infty)$ if $m = n$, in place of $[m, m+1]$. An isotopy of A will change the word by a sequence of moves of the following sort:

a) the introduction or deletion of cancelling pairs of the form $I_m I_m^{-1}$ or $I_m^{-1} I_m$,

b) left multiplication by a word in I_{k-1}, I_k and

c) right multiplication by a word in I_{l-1}, I_l.

Let $w(A)$ be the word in the free group on the symbols $I_0, I_1, \ldots, I_n$ obtained by deleting all cancelling pairs, all initial segments in I_{k-1}, I_k and all final segments in I_{l-1}, I_l. Then $w(A)$ is an isotopy invariant, and it is routine to check that A can be isotoped to read off exactly the word $w(A)$. Note that the exponents ± 1 of symbols in $w(A)$ necessarily alternate.

The action of σ_j on the word $w(A)$ is as follows, in the case that the ends of A are not in the set $\{j, j+1\}$:

$$\begin{aligned} I_m^{\pm 1} &\to I_m^{\pm 1} \qquad\qquad \text{if} \quad m \neq j\,, \\ I_j &\to I_{j-1} I_j^{-1} I_{j+1}\,, \\ I_j^{-1} &\to I_{j+1}^{-1} I_j I_{j-1}^{-1}\,. \end{aligned}$$

If an end of A happens to be $j-1$ or $j+2$, one may also have to delete an initial or final $I_{j-1}^{\pm 1}$ or $I_{j+1}^{\pm 1}$, after applying the above transformation.

Although not needed in our proof of Theorem 2.2, the next lemma will be useful later.

3.1 LEMMA. *If A is a (k, l)-arc, with $\{k, l\} \cap \{j, j+1\} = \emptyset$, such that $A * \sigma_j = A$, then up to isotopy A is disjoint from $[j, j+1]$.*

Proof. It suffices to show that $w(A)$, if reduced, does not contain $I_j^{\pm 1}$. It follows from the above rules that each occurrence of I_j in $w(A)$ is replaced by exactly one occurrence with opposite sign in $w(A * \sigma_j)$, and if we are to have $w(A) = w(A * \sigma_j)$ there will be no cancellations among the I_j in $w(A * \sigma_j)$. So if I_j occurs, we conclude $w(A) \neq w(A * \sigma_j)$, contradicting $A * \sigma_j = A$. $\square$

3.2 LEMMA. *If A is a $(j, j+1)$-arc such that $A * \sigma_j^r = A$ for some integer $r \neq 0$, then up to isotopy $A = [j, j+1]$.*

Proof. Noting that $A * \sigma_j^r = A$ if and only if $A * \sigma_j^{-r} = A$, we assume, without loss of generality, that $r > 0$. By iteration we have $A * \sigma_j^{2r} = A$. The lemma will follow if we can show that $w(A)$ must reduce to the empty word. So we suppose (for contradiction) that $w(A)$ is nonempty. First, note that then $w(A)$ must involve some symbol I_p with $|p - j| \geqslant 2$. (For otherwise $A \subset \mathbf{C} - \{(-\infty, j-1] \cup [j+2, +\infty)\}$, which is homeomorphic with $\mathbf{C}$ itself; but it is well-known that any two arcs in $\mathbf{C}$ are isotopic with fixed ends, and we would have A isotopic to $[j, j+1]$ and $w(A)$ empty.)

We assume the first and last symbols of $w(A)$ have exponent $+1$ (the other three cases can be argued similarly, or follow by symmetry). Then, referring to Figure 4, we have:

$$w(A * \sigma_j^{2r}) = (I_{j+1} I_{j-1}^{-1})^r w^* (I_{j+1}^{-1} I_{j-1})^{-r}$$

where w^* is the transformation of $w(A)$ according to the rules $(*)$ above, iterated $2r$ times. Noting that I_p persists in w^* it is easy to argue that $w(A * \sigma_j^{2r}) = w(A)$ is impossible; the contradiction. $\square$

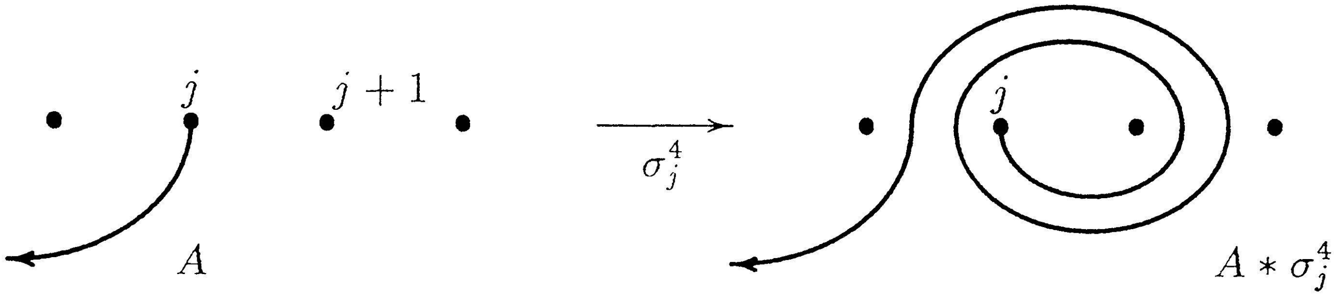


FIGURE 4

The action of $*\sigma_j^{2r}$ on a (j, k)-arc in case $r = 2$

We now turn to the proof of Theorem 2.2. It has already been observed that $(e) \Rightarrow (d) \Rightarrow (a)$, and it is obvious that $(a) \Rightarrow (c) \Rightarrow (b)$. So it remains to establish that $(b) \Rightarrow (e)$. Thus we assume that, for some $r \neq 0$, $\sigma_j^r\beta = \beta\sigma_k^r$. Since the algebraic crossing number of any two strings of a braid is a well-defined braid invariant, this equation is possible only if $\{j, j+1\} * \beta = \{k, k+1\}$. Now, noting that $\beta^{-1}\sigma_j^r\beta = \sigma_k^r$ and that σ_k^r has a (k, k)-band, we conclude that there is a proper ribbon for $\beta^{-1}\sigma_j^r\beta$ from $[k, k+1] \times 0$ to $[k, k+1] \times 1$. Define $A = \beta * [k, k+1] = [k, k+1] * \beta^{-1}$. Then we may assume (possibly after an isotopy) that the planes $\mathbf{C} \times 1/3$ and $\mathbf{C} \times 2/3$ cut the ribbon in the arcs $A \times 1/3$ and $A \times 2/3$. Moreover, the middle third of the ribbon, and Proposition 1.1, imply that $A * \sigma_j^r = A$. By Lemma 3.2, $A = [j, j+1]$ and the theorem is proved. $\square$

4. CENTRALISERS OF BRAID SUBGROUPS

We have established the following.

4.1 THEOREM. *The centraliser in* B_n *of the generator* σ_j *is the subgroup of all braids which have* (j, j)*-bands. This subgroup is isomorphic to* $B_{n-1}^j \times \mathbf{Z}$ *where* B_{n-1}^j *is the subgroup of* B_{n-1} *consisting of all* $(n-1)$*-braids whose permutations stabilise* j. $\square$

The goal of this section is to describe the centraliser of B_r in B_n, $r \leqslant n$, which we will call $C(r, n)$. Here B_r is the r-string braid group with its usual inclusion in B_n, namely as the subgroup generated by $\sigma_1 \dots \sigma_{r-1}$.

4.2 THEOREM. *The centraliser* $C(r, n)$ *of* B_r *in* B_n *consists of all* n*-braids in which the first* r *strings lie on a ribbon, disjoint from the other strings, and which intersects* $\mathbf{C} \times 0$ *and* $\mathbf{C} \times 1$ *in exactly the straight line intervals from* $[1, r] \times 0$ *and* $[1, r] \times 1$ *(up to isotopy).*

Proof. A braid β is in $C(r, n)$ if and only if it commutes with each σ_j, $1 \leqslant j \leqslant r - 1$. Thus $[j, j+1] * \beta = [j, j+1]$, $1 \leqslant j \leqslant r - 1$ and so $[1, r] * \beta = [1, r]$, up to isotopy fixing $\{1, ..., n\}$. $\square$

It follows that $C(r, n)$ consists of all n-braids constructible as follows. Let $k = n - r + 1$ and consider the subgroup B_k^1 of k-braids whose associated permutation fixes 1. Then replace the first string of a braid in B_k^1 by r parallel strings lying on a ribbon along that string. The ribbon may be twisted by some integral multiple of 2π (or π in the case $r = 2$); such braids are precisely the central elements of B_r.

4.3 THEOREM. *The centraliser* $C(r, n)$ *is isomorphic to the direct product* $B_{n-r+1}^1 \times \mathbf{Z}$. $\square$

A PRESENTATION OF $C(r, n)$. In order to establish a set of generators and defining relations for $C(r, n)$ we recall results of Chow [Ch] regarding B_k^1. This subgroup of B_k is generated by $\sigma_2, ..., \sigma_{k-1}$, together with elements $a_2, ..., a_k$ defined by

$$a_i := \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{i-2}^{-1} \sigma_{i-1}^2 \sigma_{i-2} \cdots \sigma_2 \sigma_1 .$$

These generators satisfy the usual braid relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i - j| > 1$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

as well as the following, for $i = 2, ..., k - 1$:

$$\sigma_i a_j \sigma_i^{-1} = a_j, \qquad j \neq i, i+1$$

$$\sigma_i a_i \sigma_i^{-1} = a_{i+1}$$

$$\sigma_i a_{i+1} \sigma_i^{-1} = a_{i+1}^{-1} a_i a_{i+1} .$$

In fact these are *defining* relations for B_k^1. Chow also noted that the subgroup of B_k^1 generated by the a_i is a normal subgroup (as is clear from the above relations), in fact a *free* group on the generators a_i, and that B_k^1 could be regarded as the semidirect product of that free subgroup with the subgroup generated by $\sigma_2 \ldots \sigma_{k-1}$, the latter group clearly isomorphic with the braid group on $k-1$ strings.

Applying this to our situation, for each $i = 1, \ldots, n-r$, let A_{r+i} be the n-braid resulting from replacing the first string of the k-braid a_i, defined above, by r parallel strings which lie on an untwisted band. Specifically,

$$\begin{aligned} A_{r+i} = {} & (\sigma_r^{-1}\sigma_{r+1}^{-1} \cdots \sigma_{r+i-2}^{-1}\sigma_{r+i-1})(\sigma_{r-1}^{-1}\sigma_r^{-1} \cdots \sigma_{r+i-3}^{-1}\sigma_{r+i-2}) \\ & \cdots (\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{i-1}^{-1}\sigma_i) \times (\sigma_i\sigma_{i-1} \cdots \sigma_1)(\sigma_{i+1}\sigma_i \cdots \sigma_2) \\ & \cdots (\sigma_{r+i-1}\sigma_{r+i-2} \cdots \sigma_r) \end{aligned}$$

Also let C denote the well-known generator of the centre of the r-string braid group, namely $C = \sigma_1$ if $r = 2$ and in case $r > 2$:

$$C = (\sigma_1\sigma_2 \cdots \sigma_{r-1})^r .$$

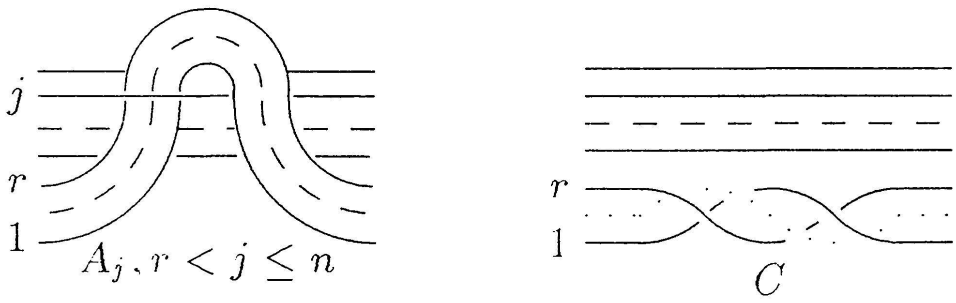


FIGURE 5

Special generators of $C(r, n)$

4.4 THEOREM. *The centraliser* $C(r,n)$ *of* B_r *in* B_n *has the generators:*

$$\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_{n-1}, A_{r+1}, \ldots, A_n, C$$

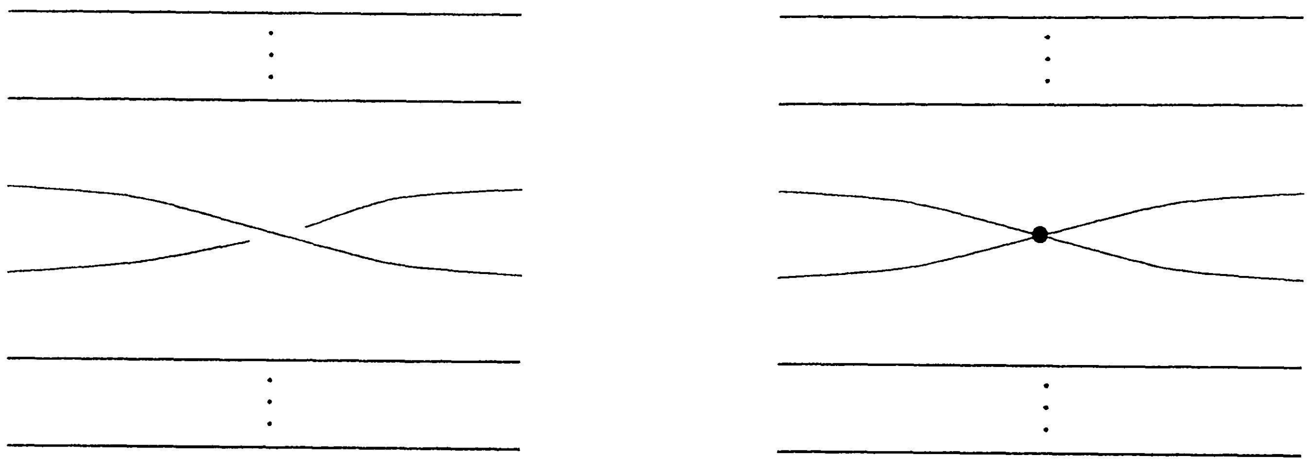
and defining relations:

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i, \qquad |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i A_j \sigma_i^{-1} &= A_j, \qquad j \neq i, i+1 \\ \sigma_i A_i \sigma_i^{-1} &= A_{i+1} \\ \sigma_i A_{i+1} \sigma_i^{-1} &= A_{i+1}^{-1} A_i A_{i+1} \\ C\sigma_i &= \sigma_i C \\ CA_i &= A_i C . \end{aligned}$$

(Subscripts ranging over all values for which the symbols are in the list of generators.) □

5. THE SINGULAR BRAID MONOID AND THE MAP η

SINGULAR BRAIDS. Just as Vassiliev [Vas] used singular knots to extend and organize knot invariants, it is useful, as in [Bae], [Bir2] to extend the group of braids to the *monoid* of singular braids. (A related construction is given in [FRR].) The strings of a singular braid are allowed to intersect, but only in discrete double points, at which they define a unique common tangent plane. As with braids, one identifies singular braids which are isotopic. The isotopy need not preserve levels, but one must move only through singular braids which have monotone strings, and the tangent plane defined by the two strings at a double point is required to vary smoothly (in 3-space) during any isotopy of the singular braids. Multiplication is by concatenation as with braids; a braid with one or more singularities is not invertible in the monoid. Let SB_n denote the monoid of singular braids on n strings; generators for SB_n are shown in Figure 6.



σ_j τ_j

FIGURE 6
Generators of SB_n

In addition to the braid generators $\sigma_1, \cdots, \sigma_{n-1}$ we have the corresponding elementary singular braids $\tau_1, \cdots, \tau_{n-1}$. Together these generate SB_n. A proof is sketched in [Bir2] that, with the invertibility of the σ_i, and the braid relations given in Section 1, the following additional relations serve to define SB_n as a monoid:

$$\sigma_i\tau_i = \tau_i\sigma_i;$$
$$\sigma_i\tau_j = \sigma_j\tau_i, \quad \tau_i\tau_j = \tau_j\tau_i, \qquad |i-j| \geqslant 2;$$
$$\sigma_i\sigma_j\tau_i = \tau_j\sigma_i\sigma_j, \qquad |i-j| = 1\,.$$

Notice that the string labels involving a particular singularity are invariant under these relations. Any equivalence between singular braids must match the first singularity involving strings i and j in one braid with the corresponding singularity of the other, etc.

5.1 PROPOSITION. *Left and right cancellation hold in SB_n, that is either of $xy = xz$ or $yx = zx$ with x, y, z in SB_n implies $y = z$.*

Proof. By symmetry and induction, it is enough to check left cancellation, and the special cases $x = \sigma_j$, which is trivial, or $x = \tau_j$. But if $\tau_j y = \tau_j z$, the singularity of τ_j in each of the two singular braids can be topologically characterised as the one involving the j and $j+1$ strings that is nearest to the left end, so an equivalence taking $\tau_j y$ to $\tau_j z$ must take τ_j to τ_j and therefore y to z. $\square$

Let us take $\mathbf{Z}B_n$ to be the group ring of the braid group B_n. Then the natural map $B_n \to \mathbf{Z}B_n$ can be extended to a monoid homomorphism, $\eta : SB_n \to \mathbf{Z}B_n$ by taking

$$\eta(\sigma_i) = \sigma_i, \ \eta(\tau_i) = \sigma_i - \sigma_i^{-1} .$$

Note that SB_n is a (2-sided) $\mathbf{Z}B_n$-module and η is a $\mathbf{Z}B_n$-homomorphism.

J. Birman in [Bir2] used this homomorphism (with target $\mathbf{C}B_n$) to establish a relation between the Vassiliev knot invariants and quantum group (or generalised Jones) invariants. She conjectured that the kernel of η is trivial, i.e. a nontrivial singular braid in the monoid SB_n never maps to zero in $\mathbf{Z}B_n$. A stronger conjecture is that η is an embedding. The weak version of Birman's conjecture (as actually stated) is rather easy — we give the proof below. The injectivity question seems much harder, and is still an open question at the time of this writing. In the next section we will apply techniques developed in the previous sections to show that η is injective at least when restricted to singular braids with no more than two singularities. We understand that Birman also has obtained these results independently.

To analyse the η map, it is useful to consider the *degree* of a (singular) braid, by which we mean the total exponent sum of all the σ_j in an expression of the braid in terms of generators. It is well-known, and easy to see from the homogeneity of the braid relations, that degree is well-defined. Likewise, the number of singularities is invariant and we define $SB_n^{(t)} \subset SB_n$ as the subset of singular braids with exactly t singularities. The following is routine to verify.

5.2 PROPOSITION. *Suppose* $x \in SB_n^{(t)}$ *is a singular braid of degree* s. *Then* $\eta(x) \in \mathbf{Z}B_n$ *is a linear combination of* 2^t *elements of* B_n *(call them terms). There is a unique term of maximal degree* $s+t$ *and a unique term of minimal degree* $s-t$. *More generally, for each integer* $u, 0 \leqslant u \leqslant t, \eta(x)$ *has* $\binom{t}{u}$ *terms of degree* $s+t-2u$, *and each of these terms has coefficient* $(-1)^u$. $\square$

There may be some cancellation among the terms of degree strictly between $s-t$ and $s+t$, but since there is only one term of maximum and one term of minimal degree, they cannot be cancelled and we draw the following conclusions.

5.3 COROLLARY. *No element of* SB_n *maps to zero under* η. $\square$

The kernel of η is also trivial in another sense.

5.4 COROLLARY. *If* $1 \in B_n \subset SB_n$ *denotes the identity braid, then* $\eta^{-1}(1) = 1$. $\square$

To close this section we consider the natural extension of η to the monoid ring $\mathbf{Z}SB_n$.

5.5 PROPOSITION. *The extension* $\eta : \mathbf{Z}SB_n \to \mathbf{Z}B_n$ *is not injective.*

Proof. τ_1 and $\sigma_1 - \sigma_1^{-1}$ are two elements of $\mathbf{Z}SB_n$ with the same image. For a more subtle example, consider the elements

$$x = \tau_1\tau_2\sigma_1^{-1} + \tau_1\sigma_2\tau_1, \quad y = \tau_2\sigma_1^{-1}\tau_2 + \sigma_2\tau_1\tau_2 .$$

An easy calculation verifies that $\eta(x) = \eta(y)$. However, $x \neq y$, as can be seen by examining their images under the map $\tau_i \to \sigma_i, \sigma_i \to \sigma_i$. $\square$

The above example is related to certain canonical relations obeyed by the Vassiliev invariants — see [Bir2], p. 274, or [Bar].

6. RESULTS REGARDING INJECTIVITY OF η

Note that if $x, y \in SB_n$ satisfy $\eta(x) = \eta(y)$, then they both have the same number of singularities, i.e. $x \in SB_n^{(t)}$ if and only if $y \in SB_n^{(t)}$. The relevance of bands to the injectivity question will be illustrated by first checking injectivity of η restricted to $SB_n^{(1)}$. (Of course, it is injective on $SB_n^{(0)} = B_n$, because it is simply the inclusion of the basis of $\mathbf{Z}B_n$.)

6.1 LEMMA. *For a braid* $\beta \in B_n$, *the following are equivalent:*

(a) $\tau_i \beta = \beta \tau_j$,

(b) $\tau_i^m \beta = \beta \tau_j^m$ *for some positive integer* m.

(c) β *has an* (i, j)*-band.*

Proof. Clearly (a) $\Rightarrow$ (b) and, using the homomorphism $SB_n \to B_n$ defined by $\tau_k \to \sigma_k$, $\sigma_k \to \sigma_k$, we see that (b) implies $\sigma_i^m \beta = \beta \sigma_j^m$, which implies (c) by Theorem 2.2. Finally, (c) $\Rightarrow$ (a), because the band can be used to convey τ_i on the left of β to become τ_j on the right. □

In Section 7 we will prove a generalisation of this lemma in which β is allowed to be a singular braid.

6.2 THEOREM. *If* $x, y \in SB_n^{(1)}$ *and* $\eta(x) = \eta(y)$, *then* $x = y$.

Proof. We can write $x = \alpha\tau_i\beta$ and $y = \alpha'\tau_j\beta'$ for (nonsingular) braids $\alpha, \alpha', \beta, \beta'$ and compute:

$$\eta(x) = \alpha\sigma_i\beta - \alpha\sigma_i^{-1}\beta\,,$$
$$\eta(y) = \alpha'\sigma_j\beta' - \alpha'\sigma_j^{-1}\beta'\,.$$

Equating the terms of highest and lowest degree, we have:

$$\alpha\sigma_i\beta = \alpha'\sigma_j\beta' \quad \text{and} \quad \alpha\sigma_i^{-1}\beta = \alpha'\sigma_j^{-1}\beta'\,.$$

It follows that

$$\sigma_i^2(\beta\beta'^{-1}) = (\beta\beta'^{-1})\sigma_j^2$$

and, by the lemma,

$$\tau_i(\beta\beta'^{-1}) = (\beta\beta'^{-1})\tau_j,$$
$$\sigma_i(\beta\beta'^{-1}) = (\beta\beta'^{-1})\sigma_j\,.$$

We quickly deduce that $\beta\beta'^{-1} = \alpha\alpha'^{-1}$ and it follows that

$$\alpha\tau_i\beta = \alpha'\tau_j\beta' \qquad \square$$

We will now work towards the injectivity of η on $SB_n^{(2)}$. Define a *singular ribbon* to be a map $R: \mathbf{I} \times \mathbf{I} \to \mathbf{C} \times \mathbf{I}$ such that R embeds $\mathbf{I} \times t$ into $\mathbf{C} \times t$, except for finitely many points t, for which the image is a single point in $\mathbf{C} \times t$. One also assumes, at these singular points, that there is a tangent plane in $\mathbf{C} \times \mathbf{I}$ for the singular ribbon. Singular ribbons are the best one can do for ribbons for singular braids. As with braids, we say a singular ribbon is *proper* for a singular braid if it sends $\{0, 1\} \times \mathbf{I}$ along two of its strands and the image is disjoint from the other strings of the singular braid. An isotopy of a singular braid can be extended to an isotopy of any of its proper singular ribbons, with the following caveat: under the equivalence $\tau_i\tau_j = \tau_j\tau_i$ one may have to reparametrise the singular ribbon.

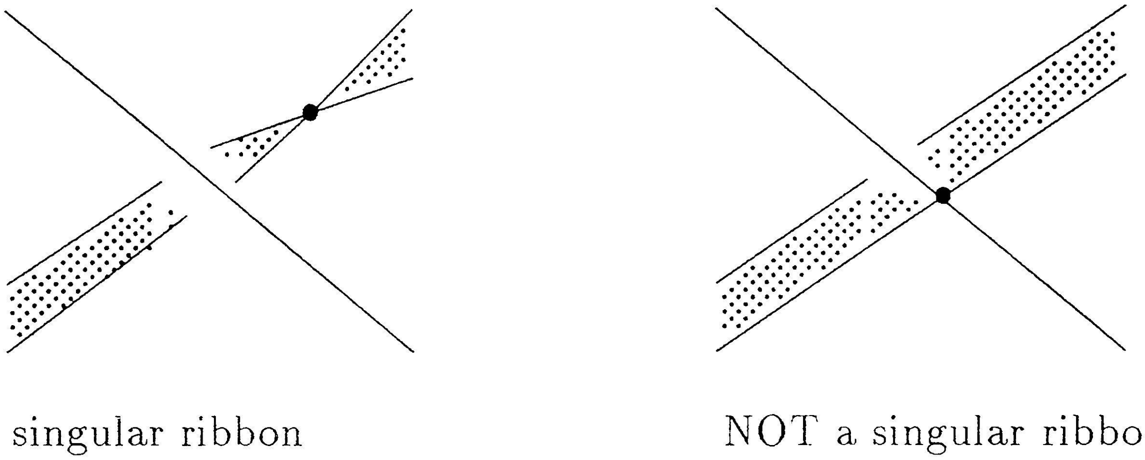


FIGURE 7

Singular ribbons only intersect two strands of a singular braid

In contrast to the situation for ordinary braids, it is not always possible to find a singular ribbon proper for a given singular braid x and with a given arc A as its intersection with $\mathbf{C} \times 0$. For example, consider an $(i, i+1)$-arc A, suppose β is a braid such that $\{i, i+1\} * \beta = \{j, j+1\}$ and consider a singular braid x of the form $x = \beta\tau_j \cdots$. Then a necessary condition for the existence of a singular ribbon, whose intersection with $\mathbf{C} \times 0$ is A, would be $A * \beta = [j, j+1]$. On the other hand, for the same reason as for ribbons, we do have the following.

6.3 PROPOSITION. *If a singular ribbon R is proper for the singular braid x and $R(\mathbf{I} \times 0)$ and $R(\mathbf{I} \times 1)$ are isotopic as proper arcs to $[j, j+1] \times 0$ and $[k, k+1] \times 1$, respectively, then $\sigma_i x = x\sigma_j$ in SB_n.* $\square$

DEFINITION. We will extend our previous definition and say that a singular braid *has a (j, k)-band* if it has a proper ribbon or singular ribbon connecting $[j, j+1] \times 0$ to $[k, k+1] \times 1$. The crucial facts we've proved are that a braid β has a (j, k)-band if and only if $\sigma_j\beta = \beta\sigma_k$, and for singular braids, having a (j, k)-band is a sufficient condition for satisfying such an equation.

6.4 LEMMA. *Let* α, β *be braids such that both* $\alpha\sigma_i\beta$ *and* $\alpha\beta$ *have* (j, k)*-bands. Then* $\alpha\tau_i\beta$ *also has a* (j, k)*-band.*

Proof. Consideration of the induced permutation implies that the pair $\{j, j+1\} * \alpha$ is either $\{i, i+1\}$ (case 1) or disjoint from $\{i, i+1\}$ (case 2). In either case, let $A = [j, j+1] * \alpha$. Then, since $\alpha\beta$ has a (j, k)-band we have $[j, j+1] * (\alpha\beta) = [k, k+1]$, and so $A = [k, k+1] * \beta^{-1} = \beta * [k, k+1]$. Similarly the hypothesis that $\alpha\sigma_i\beta$ has a (j, k)-band implies that $A * \sigma_i = A$.

Now, in case 1, A is an $(i, i+1)$-arc and we must have $A * \sigma_i = \bar{A}$. Lemma 3.2 implies that $A = [i, i+1]$. We conclude that α has a (j, i)-band and β has an (i, k)-band, and these combine with the obvious singular (i, i)-band for τ_i to provide a (j, k)-band for $\alpha\tau_i\beta$.

In case 2, Lemma 3.1 applies, and we may assume after an isotopy of the (j, k) band for $\alpha\beta$ that its intersection, A, with $C \times 1/2$ is disjoint from $[i, i+1]$. This implies that we may insert τ_i between α and β so that the singular strands are disjoint from the band, and we conclude that $\alpha\tau_i\beta$ has a nonsingular (j, k)-band. $\square$

6.5 THEOREM. *The map* η *is injective on* $SB_n^{(2)}$.

Proof. Consider an equation of the form

$$\eta(\alpha\tau_i\beta\tau_j\gamma) = \eta(\alpha'\tau_{i'}\beta'\tau_{j'}\gamma')$$

where $\alpha, \alpha', \beta, \beta', \gamma, \gamma', \in B_n$.

Now

$$\eta(\alpha\tau_i\beta\tau_j\gamma) = \alpha\sigma_i\beta\sigma_j\gamma - \alpha\sigma_i^{-1}\beta\sigma_j\gamma - \alpha\sigma_i\beta\sigma_j^{-1}\gamma + \alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma$$

and $\eta(\alpha'\tau_{i'}\beta'\tau_{j'}\gamma')$ has a similar expansion. If they are equal in $\mathbf{Z}B_n$, then considering the degrees we must have one of two sets of equations. Either

$$\alpha\sigma_i\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}\gamma' \tag{1}$$

$$\alpha\sigma_i^{-1}\beta\sigma_j\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}\gamma' \tag{2}$$

$$\alpha\sigma_i\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}^{-1}\gamma' \tag{3}$$

$$\alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}^{-1}\gamma' \tag{4}$$

or

$$\alpha\sigma_i\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}\gamma' \tag{1}$$

$$\alpha\sigma_i^{-1}\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}^{-1}\gamma' \tag{2'}$$

$$\alpha\sigma_i\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}\gamma' \tag{3'}$$

$$\alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}^{-1}\gamma' \tag{4}$$

We claim that in either case the following are true:

$$\alpha\beta\gamma = \alpha'\beta'\gamma' \tag{5}$$

$$\alpha\tau_i\beta\tau_j\gamma = \alpha'\tau_{i'}\beta'\tau_{j'}\gamma' . \tag{6}$$

Assume initially that (1), (2), (3) and (4) are satisfied. Eliminating $\beta'\sigma_{j'}\gamma'$ between (1) and (2) gives $\alpha'^{-1}\alpha\sigma_i^2 = \sigma_{i'}^2\alpha'^{-1}\alpha$. The main theorem now implies that $\alpha'^{-1}\alpha$ has an (i', i)-band. Similarly eliminating $\alpha'\sigma_{i'}\beta'$ between (1) and (3) implies that $\gamma\gamma'^{-1}$ has a (j, j')-band. Applying these facts to (1) gives

$$\sigma_{i'}\beta'\sigma_{j'} = \alpha'^{-1}\alpha\sigma_i\beta\sigma_j\gamma\gamma'^{-1} = \sigma_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\sigma_{j'}$$

and (5) follows in this case.

Similarly using (5)

$$\tau_{i'}\beta'\tau_{j'} = \tau_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\tau_j = \alpha'^{-1}\alpha\tau_i\beta\tau_j\gamma\gamma'^{-1} ,$$

and therefore (6) also holds in this case.

Now assume that the equations (1), (2′), (3′) and (4) hold. A similar elimination as in the first case implies that $\beta\sigma_j\gamma\gamma'^{-1}$ has an (i, j')-band and $\alpha'^{-1}\alpha\sigma_i\beta$ has an (i', j)-band. So

$$\sigma_{i'}\beta'\sigma_{j'} = \alpha'^{-1}\alpha\sigma_i\beta\sigma_j\gamma\gamma'^{-1} = \sigma_{i'}\alpha'^{-1}\alpha\sigma_i\beta\gamma\gamma'^{-1}$$

The above can be written as

$$\alpha\sigma_i\beta\gamma = \alpha'\beta'\sigma_{j'}\gamma' \tag{7}$$

Similarly from equation (4) we have

$$\alpha\sigma_i^{-1}\beta\gamma = \alpha'\beta'\sigma_{j'}^{-1}\gamma' \tag{8}$$

Eliminating $\alpha^{-1}\alpha'\beta'$ between (7) and (8) gives $\sigma_i^2\beta\gamma\gamma'^{-1} = \beta\gamma\gamma'^{-1}\sigma_{j'}^2$, so $\beta\gamma\gamma'^{-1}$ has an (i, j')-band, and with Lemma 6.6 we deduce that $\beta\tau_j\gamma\gamma'^{-1}$ has an (i, j')-band. We can also conclude that equation (5) holds in this case. A similar argument shows that $\alpha'^{-1}\alpha\beta$ has an (i', j)-band.

Hence

$$\begin{aligned} \alpha'^{-1}\alpha\tau_i\beta\tau_j\gamma\gamma'^{-1} &= \alpha'^{-1}\alpha\beta\tau_j\gamma\gamma'^{-1}\tau_{j'} && (i, j')\text{-band} \\ &= \tau_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\tau_{j'} && (i', j)\text{-band} \\ &= \tau_{i'}\beta'\tau_{j'} . \end{aligned}$$

So (6) is true in this case as well. □

7. Centralisers in SB_n

7.1 Theorem. *For a singular braid $x \in SB_n$ the following are equivalent:*

(a) $\sigma_j x = x\sigma_k$*;*

(b) $\sigma_j^r x = x\sigma_k^r$*, for some nonzero integer* r*;*

(c) $\sigma_j^r x = x\sigma_k^r$*, for every integer* r*;*

(d) $\tau_j x = x\tau_k$*;*

(e) $\tau_j^r x = x\tau_k^r$*, for some nonzero integer* r*;*

(f) x *has a (possibly singular)* (j, k)*-band.*

Proof. We only show (b) implies (f). The other implications are quite clear. The term "band" will include the possibility of a singular band. Suppose $\sigma_j^r x = x\sigma_k^r$. Then we have that $\sigma_j^{2r} x = x\sigma_k^{2r}$. Assume $x = \beta\tau_i y$, where β is a braid; in other words τ_i is the first singular generator appearing in x. Then we have $\beta^{-1}\sigma_j^{2r}\beta\tau_i y = \tau_i y\sigma_k^{2r}$. Recall that isotopy, or the extended Reidemeister moves for singular braids, do not change the order of singular generators on the same strings. Since $\beta^{-1}\sigma_j^{2r}\beta$ is a pure braid, the τ_i in $\tau_i y\sigma_k^{2r}$ corresponds under some homeomorphism, to the τ_i in $\beta^{-1}\sigma_j^{2r}\beta\tau_i y$. Hence the image, under that homeomorphism, of the trivial singular band near the first τ_i provides a band for $\beta^{-1}\sigma_j^{2r}\beta$. Therefore, τ_i commutes with $\beta^{-1}\sigma_j^{2r}\beta$. It follows that $\tau_i\beta^{-1}\sigma_j^{2r}\beta y = \tau_i y\sigma_k^{2r}$. By Proposition 5.1, we have $\beta^{-1}\sigma_j^{2r}\beta y = y\sigma_k^{2r}$, i.e. $\sigma_j^{2r}\beta y = \beta y\sigma_k^{2r}$. By induction, βy has a (j, k)-band. Since τ_i commutes with $\beta^{-1}\sigma_j^{2r}\beta$, so does σ_i, thus we have $\beta^{-1}\sigma_j^{2r}\beta\sigma_i y = \sigma_i\beta^{-1}\sigma_j^{2r}\beta y = \sigma_i y\sigma_k^{2r}$, i.e. $\sigma_j^{2r}\beta\sigma_i y = \beta\sigma_i y\sigma_k^{2r}$. It follows from induction assumption that $\beta\sigma_i y$ has a (j, k)-band. Since both βy and $\beta\sigma_i y$ have a (j, k)-band, we can use the argument of Lemma 6.4 to conclude that $x = \beta\tau_i y$ has a (j, k)-band. □

The above theorem allows us to identify monoid centralisers in SB_n. Notice that SB_2 is abelian. On the other hand, for $n \geqslant 3$, any singular braid with a singularity involving strings labelled, say, j and k, $j < k$, could not possibly commute with τ_k, as any (singular) band from $[k, k+1] \times 0$ to $[k, k+1] \times 1$ would have a forbidden intersection with the j string, see Figure 7. Therefore for $n \geqslant 3$, only *nonsingular* braids are central. We will conclude with two applications whose proofs, at this point, can safely be left to the reader.

7.2 THEOREM. *The centre of* SB_n *is all of* SB_n *for* $n = 2$. *But in case* $n \geqslant 3$ *it is the same as the (infinite cyclic) centre of* $B_n \subset SB_n$, *generated by* Δ^2. $\square$

7.3 THEOREM. *Under the natural inclusion, the centraliser of* SB_r *in* SB_n, $r \leqslant n$, *is generated as a monoid by the generators (see Theorem 4.4) of* $C(r, n)$*:*

$$\sigma_{r+1}, \sigma_{r+2}, \ldots, \sigma_{n-1}, A_{r+1}, \ldots, A_n, C,$$

together with their inverses and the singular generators:

$$\tau_{r+1}, \ldots, \tau_{n-1} \quad \text{if } r \geqslant 3, \text{ or}$$
$$\tau_1, \tau_3, \tau_4, \ldots, \tau_{n-1} \quad \text{if } r = 2. \qquad \square$$

REFERENCES

[Art 1] ARTIN, E. Theorie der Zöpfe. *Abh. Math. Sem. Hamburg Univ. 4* (1926), 47-72.

[Art 2] —— Theory of Braids. *Ann. of Math. (2) 48* (1947), 101-128.

[Bae] BAEZ, J. Link invariants and perturbation theory. *Lett. Math. Phys. 2* (1992), 43-51.

[Bar] BAR-NATAN, D. On the Vassiliev knot invariants. *Topology 34* (1995), 423-471.

[Bir1] BIRMAN, J. *Braids, links and mapping class groups.* Ann. Math. Studies 82, Princeton University Press, Princeton, N.J., 1974.

[Bir2] —— New points of view in knot theory. *Bull. AMS 28* (1993), 253-287.

[Bur] BURDE, G. Über Normalisatoren der Zopfgruppe. *Abh. Math. Sem. Hamburg 27* (1964), 97-115.

[BZ] BURDE, G. and H. ZIESCHANG. *Knots.* De Gruyter Studies in Math. 5, New York, 1985.

[Char1] CHARNEY, R. Artin groups of finite type are biautomatic. *Math. Ann. 292* (1992), 671-693.

[Char2] —— Geodesic automation and growth functions for Artin groups of finite type. *Math. Ann. 301* (1995), 307-324.

[Chow] CHOW, W.-L. On the algebraical braid group. *Ann. of Math. (2) 49* (1948), 654-658.

[ECHLPT] EPSTEIN, D., J. CANNON, D. HOLT, S. LEVY, M. PATERSON and W. THURSTON. *Word processing in groups.* Jones and Bartlett, 1992.

[FRR] FENN, R., R. RIMÁNYI and C. ROURKE. The braid-permutation group. Preprint (submitted for publication).

[Gar] GARSIDE, F. The braid group and other groups. *Quart. J. Math. Oxford Ser. (2) 20* (1969), 235-254.

[GS] GERSTEN, S. and H. SHORT. Rational subgroups of biautomatic groups. *Ann. Math. 105* (1991), 641-662.

[Gur1] GURZO, G. Centralisers of finite sets of elements of the braid group B_{n+1}. *Mat.-Zametki 37* (1985), 3-6. (English translation: *Math. Notes 37 (1985),* 3-4.)

[Gur2] —— Systems of generators for normalisers of certain elements of a braid group. *Izv. Akad. Nauk SSSR ser. Mat. 48* (1984), 476-519. (English translation: *Math. USSR Izvestiya 24* (1985), 439-478.)

[Han] HANSEN, V. *Braids and coverings: Selected topics.* London Math. Soc. Student Texts, vol. 18, Cambridge U. Press 1989.

[Mak] MAKANIN, G. S. On normalizers of the braid group. *Mat. Sb. 86* (1971), 171-179.

[Vas] VASSILIEV, V. A. Cohomology of knot spaces. *Theory of singularities and its applications* (V.I. Arnold, ed.), Amer. Math. Soc., Providence 1990.

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