

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 42 (1996)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CENTRALISERS IN THE BRAID GROUP AND SINGULAR BRAID MONOID  
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**Kapitel:** 6. Results regarding injectivity of  
**DOI:** <https://doi.org/10.5169/seals-87872>

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**5.2 PROPOSITION.** *Suppose  $x \in SB_n^{(t)}$  is a singular braid of degree  $s$ . Then  $\eta(x) \in \mathbb{Z}B_n$  is a linear combination of  $2^t$  elements of  $B_n$  (call them terms). There is a unique term of maximal degree  $s + t$  and a unique term of minimal degree  $s - t$ . More generally, for each integer  $u$ ,  $0 \leq u \leq t$ ,  $\eta(x)$  has  $\binom{t}{u}$  terms of degree  $s + t - 2u$ , and each of these terms has coefficient  $(-1)^u$ .  $\square$*

There may be some cancellation among the terms of degree strictly between  $s - t$  and  $s + t$ , but since there is only one term of maximum and one term of minimal degree, they cannot be cancelled and we draw the following conclusions.

**5.3 COROLLARY.** *No element of  $SB_n$  maps to zero under  $\eta$ .  $\square$*

The kernel of  $\eta$  is also trivial in another sense.

**5.4 COROLLARY.** *If  $1 \in B_n \subset SB_n$  denotes the identity braid, then  $\eta^{-1}(1) = 1$ .  $\square$*

To close this section we consider the natural extension of  $\eta$  to the monoid ring  $\mathbb{Z}SB_n$ .

**5.5 PROPOSITION.** *The extension  $\eta: \mathbb{Z}SB_n \rightarrow \mathbb{Z}B_n$  is not injective.*

*Proof.*  $\tau_1$  and  $\sigma_1 - \sigma_1^{-1}$  are two elements of  $\mathbb{Z}SB_n$  with the same image. For a more subtle example, consider the elements

$$x = \tau_1 \tau_2 \sigma_1^{-1} + \tau_1 \sigma_2 \tau_1, \quad y = \tau_2 \sigma_1^{-1} \tau_2 + \sigma_2 \tau_1 \tau_2.$$

An easy calculation verifies that  $\eta(x) = \eta(y)$ . However,  $x \neq y$ , as can be seen by examining their images under the map  $\tau_i \rightarrow \sigma_i, \sigma_i \rightarrow \sigma_i$ .  $\square$

The above example is related to certain canonical relations obeyed by the Vassiliev invariants — see [Bir2], p. 274, or [Bar].

## 6. RESULTS REGARDING INJECTIVITY OF $\eta$

Note that if  $x, y \in SB_n$  satisfy  $\eta(x) = \eta(y)$ , then they both have the same number of singularities, i.e.  $x \in SB_n^{(t)}$  if and only if  $y \in SB_n^{(t)}$ . The relevance of bands to the injectivity question will be illustrated by first checking

injectivity of  $\eta$  restricted to  $SB_n^{(1)}$ . (Of course, it is injective on  $SB_n^{(0)} = B_n$ , because it is simply the inclusion of the basis of  $\mathbb{Z}B_n$ .)

6.1 LEMMA. *For a braid  $\beta \in B_n$ , the following are equivalent:*

- (a)  $\tau_i \beta = \beta \tau_j$ ,
- (b)  $\tau_i^m \beta = \beta \tau_j^m$  for some positive integer  $m$ .
- (c)  $\beta$  has an  $(i, j)$ -band.

*Proof.* Clearly (a)  $\Rightarrow$  (b) and, using the homomorphism  $SB_n \rightarrow B_n$  defined by  $\tau_k \rightarrow \sigma_k$ ,  $\sigma_k \rightarrow \sigma_k$ , we see that (b) implies  $\sigma_i^m \beta = \beta \sigma_j^m$ , which implies (c) by Theorem 2.2. Finally, (c)  $\Rightarrow$  (a), because the band can be used to convey  $\tau_i$  on the left of  $\beta$  to become  $\tau_j$  on the right.  $\square$

In Section 7 we will prove a generalisation of this lemma in which  $\beta$  is allowed to be a singular braid.

6.2 THEOREM. *If  $x, y \in SB_n^{(1)}$  and  $\eta(x) = \eta(y)$ , then  $x = y$ .*

*Proof.* We can write  $x = \alpha \tau_i \beta$  and  $y = \alpha' \tau_j \beta'$  for (nonsingular) braids  $\alpha, \alpha', \beta, \beta'$  and compute:

$$\begin{aligned}\eta(x) &= \alpha \sigma_i \beta - \alpha \sigma_i^{-1} \beta, \\ \eta(y) &= \alpha' \sigma_j \beta' - \alpha' \sigma_j^{-1} \beta' .\end{aligned}$$

Equating the terms of highest and lowest degree, we have:

$$\alpha \sigma_i \beta = \alpha' \sigma_j \beta' \quad \text{and} \quad \alpha \sigma_i^{-1} \beta = \alpha' \sigma_j^{-1} \beta' .$$

It follows that

$$\sigma_i^2 (\beta \beta'^{-1}) = (\beta \beta'^{-1}) \sigma_j^2$$

and, by the lemma,

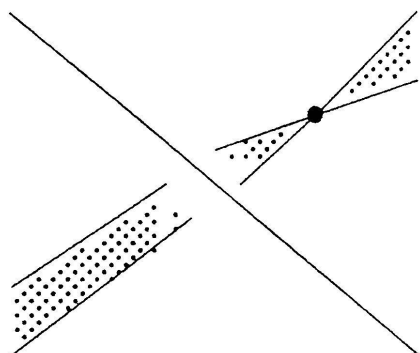
$$\begin{aligned}\tau_i (\beta \beta'^{-1}) &= (\beta \beta'^{-1}) \tau_j, \\ \sigma_i (\beta \beta'^{-1}) &= (\beta \beta'^{-1}) \sigma_j .\end{aligned}$$

We quickly deduce that  $\beta \beta'^{-1} = \alpha \alpha'^{-1}$  and it follows that

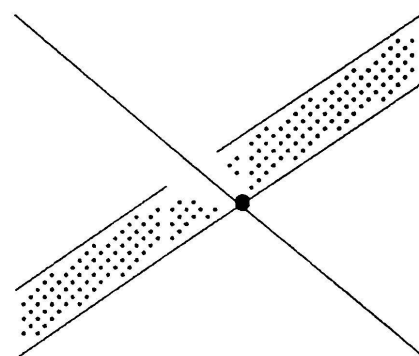
$$\alpha \tau_i \beta = \alpha' \tau_j \beta' \quad \square$$

We will now work towards the injectivity of  $\eta$  on  $SB_n^{(2)}$ . Define a *singular ribbon* to be a map  $R: \mathbf{I} \times \mathbf{I} \rightarrow \mathbf{C} \times \mathbf{I}$  such that  $R$  embeds  $\mathbf{I} \times t$  into  $\mathbf{C} \times t$ , except for finitely many points  $t$ , for which the image is a single point in  $\mathbf{C} \times t$ . One also assumes, at these singular points, that there is a

tangent plane in  $\mathbf{C} \times \mathbf{I}$  for the singular ribbon. Singular ribbons are the best one can do for ribbons for singular braids. As with braids, we say a singular ribbon is *proper* for a singular braid if it sends  $\{0, 1\} \times \mathbf{I}$  along two of its strands and the image is disjoint from the other strings of the singular braid. An isotopy of a singular braid can be extended to an isotopy of any of its proper singular ribbons, with the following caveat: under the equivalence  $\tau_i \tau_j = \tau_j \tau_i$  one may have to reparametrise the singular ribbon.



A singular ribbon



NOT a singular ribbon

FIGURE 7

Singular ribbons only intersect two strands of a singular braid

In contrast to the situation for ordinary braids, it is not always possible to find a singular ribbon proper for a given singular braid  $x$  and with a given arc  $A$  as its intersection with  $\mathbf{C} \times 0$ . For example, consider an  $(i, i + 1)$ -arc  $A$ , suppose  $\beta$  is a braid such that  $\{i, i + 1\} * \beta = \{j, j + 1\}$  and consider a singular braid  $x$  of the form  $x = \beta \tau_j \cdots$ . Then a necessary condition for the existence of a singular ribbon, whose intersection with  $\mathbf{C} \times 0$  is  $A$ , would be  $A * \beta = [j, j + 1]$ . On the other hand, for the same reason as for ribbons, we do have the following.

**6.3 PROPOSITION.** *If a singular ribbon  $R$  is proper for the singular braid  $x$  and  $R(\mathbf{I} \times 0)$  and  $R(\mathbf{I} \times 1)$  are isotopic as proper arcs to  $[j, j + 1] \times 0$  and  $[k, k + 1] \times 1$ , respectively, then  $\sigma_i x = x \sigma_j$  in  $SB_n$ .  $\square$*

**DEFINITION.** We will extend our previous definition and say that a singular braid *has a  $(j, k)$ -band* if it has a proper ribbon or singular ribbon connecting  $[j, j + 1] \times 0$  to  $[k, k + 1] \times 1$ . The crucial facts we've proved are that a braid  $\beta$  has a  $(j, k)$ -band if and only if  $\sigma_j \beta = \beta \sigma_k$ , and for singular braids, having a  $(j, k)$ -band is a sufficient condition for satisfying such an equation.

6.4 LEMMA. *Let  $\alpha, \beta$  be braids such that both  $\alpha\sigma_i\beta$  and  $\alpha\beta$  have  $(j, k)$ -bands. Then  $\alpha\tau_i\beta$  also has a  $(j, k)$ -band.*

*Proof.* Consideration of the induced permutation implies that the pair  $\{j, j+1\} * \alpha$  is either  $\{i, i+1\}$  (case 1) or disjoint from  $\{i, i+1\}$  (case 2). In either case, let  $A = [j, j+1] * \alpha$ . Then, since  $\alpha\beta$  has a  $(j, k)$ -band we have  $[j, j+1] * (\alpha\beta) = [k, k+1]$ , and so  $A = [k, k+1] * \beta^{-1} = \beta * [k, k+1]$ . Similarly the hypothesis that  $\alpha\sigma_i\beta$  has a  $(j, k)$ -band implies that  $A * \sigma_i = A$ .

Now, in case 1,  $A$  is an  $(i, i+1)$ -arc and we must have  $A * \sigma_i = \bar{A}$ . Lemma 3.2 implies that  $A = [i, i+1]$ . We conclude that  $\alpha$  has a  $(j, i)$ -band and  $\beta$  has an  $(i, k)$ -band, and these combine with the obvious singular  $(i, i)$ -band for  $\tau_i$  to provide a  $(j, k)$ -band for  $\alpha\tau_i\beta$ .

In case 2, Lemma 3.1 applies, and we may assume after an isotopy of the  $(j, k)$  band for  $\alpha\beta$  that its intersection,  $A$ , with  $C \times 1/2$  is disjoint from  $[i, i+1]$ . This implies that we may insert  $\tau_i$  between  $\alpha$  and  $\beta$  so that the singular strands are disjoint from the band, and we conclude that  $\alpha\tau_i\beta$  has a nonsingular  $(j, k)$ -band.  $\square$

6.5 THEOREM. *The map  $\eta$  is injective on  $SB_n^{(2)}$ .*

*Proof.* Consider an equation of the form

$$\eta(\alpha\tau_i\beta\tau_j\gamma) = \eta(\alpha'\tau_{i'}\beta'\tau_{j'}\gamma')$$

where  $\alpha, \alpha', \beta, \beta', \gamma, \gamma' \in B_n$ .

Now

$$\eta(\alpha\tau_i\beta\tau_j\gamma) = \alpha\sigma_i\beta\sigma_j\gamma - \alpha\sigma_i^{-1}\beta\sigma_j\gamma - \alpha\sigma_i\beta\sigma_j^{-1}\gamma + \alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma$$

and  $\eta(\alpha'\tau_{i'}\beta'\tau_{j'}\gamma')$  has a similar expansion. If they are equal in  $\mathbf{Z}B_n$ , then considering the degrees we must have one of two sets of equations. Either

$$(1) \quad \alpha\sigma_i\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}\gamma'$$

$$(2) \quad \alpha\sigma_i^{-1}\beta\sigma_j\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}\gamma'$$

$$(3) \quad \alpha\sigma_i\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}^{-1}\gamma'$$

$$(4) \quad \alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}^{-1}\gamma'$$

or

$$(1) \quad \alpha\sigma_i\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}\gamma'$$

$$(2') \quad \alpha\sigma_i^{-1}\beta\sigma_j\gamma = \alpha'\sigma_{i'}\beta'\sigma_{j'}^{-1}\gamma'$$

$$(3') \quad \alpha\sigma_i\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}\gamma'$$

$$(4) \quad \alpha\sigma_i^{-1}\beta\sigma_j^{-1}\gamma = \alpha'\sigma_{i'}^{-1}\beta'\sigma_{j'}^{-1}\gamma'$$

We claim that in either case the following are true:

$$(5) \quad \alpha\beta\gamma = \alpha'\beta'\gamma'$$

$$(6) \quad \alpha\tau_i\beta\tau_j\gamma = \alpha'\tau_{i'}\beta'\tau_{j'}\gamma'.$$

Assume initially that (1), (2), (3) and (4) are satisfied. Eliminating  $\beta'\sigma_{j'}\gamma'$  between (1) and (2) gives  $\alpha'^{-1}\alpha\sigma_i^2 = \sigma_{i'}^2\alpha'^{-1}\alpha$ . The main theorem now implies that  $\alpha'^{-1}\alpha$  has an  $(i', i)$ -band. Similarly eliminating  $\alpha'\sigma_{i'}\beta'$  between (1) and (3) implies that  $\gamma\gamma'^{-1}$  has a  $(j, j')$ -band. Applying these facts to (1) gives

$$\sigma_{i'}\beta'\sigma_{j'} = \alpha'^{-1}\alpha\sigma_i\beta\sigma_j\gamma\gamma'^{-1} = \sigma_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\sigma_{j'}$$

and (5) follows in this case.

Similarly using (5)

$$\tau_{i'}\beta'\tau_{j'} = \tau_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\tau_j = \alpha'^{-1}\alpha\tau_i\beta\tau_j\gamma\gamma'^{-1},$$

and therefore (6) also holds in this case.

Now assume that the equations (1), (2'), (3') and (4) hold. A similar elimination as in the first case implies that  $\beta\sigma_j\gamma\gamma'^{-1}$  has an  $(i, j')$ -band and  $\alpha'^{-1}\alpha\sigma_i\beta$  has an  $(i', j)$ -band. So

$$\sigma_{i'}\beta'\sigma_{j'} = \alpha'^{-1}\alpha\sigma_i\beta\sigma_j\gamma\gamma'^{-1} = \sigma_{i'}\alpha'^{-1}\alpha\sigma_i\beta\gamma\gamma'^{-1}$$

The above can be written as

$$(7) \quad \alpha\sigma_i\beta\gamma = \alpha'\beta'\sigma_{j'}\gamma'$$

Similarly from equation (4) we have

$$(8) \quad \alpha\sigma_i^{-1}\beta\gamma = \alpha'\beta'\sigma_{j'}^{-1}\gamma'$$

Eliminating  $\alpha^{-1}\alpha'\beta'$  between (7) and (8) gives  $\sigma_i^2\beta\gamma\gamma'^{-1} = \beta\gamma\gamma'^{-1}\sigma_{j'}^2$ , so  $\beta\gamma\gamma'^{-1}$  has an  $(i, j')$ -band, and with Lemma 6.6 we deduce that  $\beta\tau_j\gamma\gamma'^{-1}$  has an  $(i, j')$ -band. We can also conclude that equation (5) holds in this case. A similar argument shows that  $\alpha'^{-1}\alpha\beta$  has an  $(i', j)$ -band.

Hence

$$\begin{aligned} \alpha'^{-1}\alpha\tau_i\beta\tau_j\gamma\gamma'^{-1} &= \alpha'^{-1}\alpha\beta\tau_j\gamma\gamma'^{-1}\tau_{j'} && (i, j')\text{-band} \\ &= \tau_{i'}\alpha'^{-1}\alpha\beta\gamma\gamma'^{-1}\tau_{j'} && (i', j)\text{-band} \\ &= \tau_{i'}\beta'\tau_{j'}. \end{aligned}$$

So (6) is true in this case as well.  $\square$