2. An exhaustion of locally symmetric spaces

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2. An exhaustion of locally symmetric spaces

Let X be a Riemannian symmetric space of noncompact type and rank ≥ 2 and let Γ be a non-uniform, torsion-free lattice in the group of isometries of X. In this section we briefly describe the basic features of an exhaustion of the locally symmetric space $V = \Gamma\backslash X$ by Riemannian polyhedra, which was previously constructed in [L2].

The idea is to work with a *fundamental set* $\Omega \subset X$ for the discrete (arithmetic) group Γ. Such "coarse" fundamental domains are provided by *reduction theory*; they are finite unions of translates of so-called Siegel sets. We begin with reviewing some facts about linear algebraic groups and set up the notation. Roughly speaking, the lattice Γ determines a "$\mathbb{Q}$-structure" on the real Lie group of isometries of X such that Γ is given by integer matrices. The symmetric space X in turn inherits canonical parametrizations adopted to this structure (generalized horocyclic coordinates). Siegel sets are then defined with respect to such parametrizations.

2.1. Reduction theory and geometry at infinity

We denote by G the identity component of the group of isometries of X; it is a connected, semisimple Lie group with trivial center. We shall always assume in the following that the non-uniform lattice Γ is *irreducible* (see [R2] 5.20). Then, by the arithmeticity theorem of Margulis, there is a connected semisimple linear algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$, $\mathbb{Q}$-embedded in a general linear group $\mathbf{GL}(N,\mathbb{C})$, and a Lie group isomorphism $p : G \longrightarrow \mathbf{G}(\mathbb{R})^0$ such that $p(\Gamma)$ is *arithmetic*, i.e. $p(\Gamma) \subset \mathbf{G}(\mathbb{Q}) \subset \mathbf{GL}(N,\mathbb{C})$ is commensurable with the group $\mathbf{G}(\mathbb{Z}) = \mathbf{G} \cap \mathbf{GL}(N,\mathbb{Z})$ (see [Z] 3.1.6 and 6.1.10). The symmetric space X can be recovered as the manifold of maximal compact subgroups of the identity component of the group $\mathbf{G}(\mathbb{R}) = \mathbf{G} \cap \mathbf{GL}(N,\mathbb{R})$ of $\mathbb{R}$-rational points of $\mathbf{G}$. For simplicity we will always identify G with $\mathbf{G}(\mathbb{R})^0$ and Γ with $p(\Gamma)$.

Let $\mathbf{S}$ (resp. $\mathbf{T}$) be a maximal $\mathbb{Q}$-split (resp. $\mathbb{R}$-split) algebraic torus of $\mathbf{G}$, i.e. a subgroup of $\mathbf{G}$ which is isomorphic over $\mathbb{Q}$ (resp. $\mathbb{R}$) to the direct product of q (resp. $r \geq q$) copies of $\mathbb{C}^*$. All such tori are conjugate under $\mathbf{G}(\mathbb{Q}) = \mathbf{G} \cap \mathbf{GL}(N,\mathbb{Q})$ $\big($resp. $\mathbf{G}(\mathbb{R})\big)$ and their common dimension q (resp. r) is called the *$\mathbb{Q}$-rank* (resp. *$\mathbb{R}$-rank*) of $\mathbf{G}$. The identity component of $\mathbf{S}(\mathbb{R})$ $\big($resp. $\mathbf{T}(\mathbb{R})\big)$ will be denoted by A (resp. A_0), the corresponding Lie algebras by $\mathfrak{a}$ (resp. $\mathfrak{a}_0$). The $\mathbb{R}$-rank of $\mathbf{G}$ coincides with the rank of the symmetric space X, i.e. the maximal dimension of totally geodesic flat subspaces. The choice of a maximal compact subgroup K of G

is equivalent to the choice of a base point x_0 of X. We can choose K with Lie algebra $\mathfrak{k}$ so that under the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G we have $\mathfrak{a} \subseteq \mathfrak{a}_0 \subset \mathfrak{p} \cong T_{x_0}X$. Here $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}$, i.e. the tangent space at x_0 of the (maximal $\mathbb{R}$-) flat $A_0 \cdot x_0$ in X. The pair of Lie algebras $(\mathfrak{g}, \mathfrak{a}_0)$ gives rise to the root system ${}_{\mathbb{R}}\Phi$ of the symmetric space. Similarly there is a system of $\mathbb{Q}$-roots ${}_{\mathbb{Q}}\Phi$ associated to the pair $(\mathfrak{g}, \mathfrak{a})$ (see [B3] §21). It is always possible to choose orderings of ${}_{\mathbb{Q}}\Phi$ and ${}_{\mathbb{R}}\Phi$ such that the restrictions of simple $\mathbb{R}$-roots of ${}_{\mathbb{R}}\Phi$ to $\mathfrak{a}$ are either *simple* $\mathbb{Q}$-roots of ${}_{\mathbb{Q}}\Phi$, i.e. the elements of a basis $\Delta = {}_{\mathbb{Q}}\Delta$ of ${}_{\mathbb{Q}}\Phi$, or zero (see [BT] 6.8). The basis ${}_{\mathbb{R}}\Delta$ defines a closed $\mathbb{R}$-Weyl chamber $\overline{\mathfrak{a}_0^+}$ in $\mathfrak{a}_0$ and Δ then determines a closed *$\mathbb{Q}$-Weyl chamber* $\overline{\mathfrak{a}^+} := \{H \in \mathfrak{a} \mid \alpha(H) \geq 0, \text{ for all } \alpha \in \Delta\}$ in $\mathfrak{a}$. We set $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$ (resp. $\overline{A_0^+} = \exp \overline{\mathfrak{a}_0^+}$). A *$\mathbb{Q}$-Weyl chamber in X* is a translate of the basic chamber $\overline{A^+} \cdot x_0 \subseteq \overline{A_0^+} \cdot x_0$. The elements of Δ are differentials of characters (defined over $\mathbb{Q}$) of the maximal $\mathbb{Q}$-split torus $\mathbf{S}$. It is convenient to identify the elements of Δ also with such characters. When restricted to A their values are denoted by $\alpha(a)$ $(a \in A, \alpha \in \Delta)$. Notice that $\overline{A^+} = \{a \in A \mid \alpha(a) \geq 1 \text{ for all } \alpha \in \Delta\}$.

A closed subgroup $\mathbf{P}$ of $\mathbf{G}$ defined over $\mathbb{Q}$ is a *parabolic $\mathbb{Q}$-subgroup* if $\mathbf{G}/\mathbf{P}$ is a projective variety (see [B3] §11). A *parabolic $\mathbb{Q}$-subgroup P of* $G = \mathbf{G}(\mathbb{R})^0$ is by definition the intersection of G with a parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ (see [BS]). The conjugacy classes under $\mathbf{G}(\mathbb{Q})$ of parabolic $\mathbb{Q}$-subgroups are in one-to-one correspondence with the subsets Θ of the (chosen) set Δ of simple $\mathbb{Q}$-roots; they are represented by the *standard parabolic* $\mathbb{Q}$-subgroups $\mathbf{P}_\Theta$ of $\mathbf{G}$ (see [B3] §21.11). The corresponding standard parabolic $\mathbb{Q}$-subgroups of G are denoted by P_Θ. The minimal parabolic subgroup $P = P_\varnothing$ has a decomposition $P = UMA$, where U is unipotent and M is reductive; A centralizes M and normalizes U (see [B1]). This yields a (generalized) Iwasawa decomposition for G, i.e. $G = P \cdot K = UMAK$, which implies that P acts transitively on the symmetric space X. The intersection of the maximal compact subgroup K of G with M is maximal compact in M and the quotient $Z = M/(K \cap M)$ is (in general) the Riemannian product of a symmetric space of noncompact type by a (flat) Euclidean space. Let $\tau : M \longrightarrow Z$ be the natural projection. Then the "horocyclic coordinate map"

$$\mu : Y = U \times Z \times A \longmapsto X \quad ; \quad \big(u, \tau(m), a\big) \longmapsto uma \cdot x_0$$

is an isomorphism of analytic manifolds (see [BS] or [B2]).

A *generalized Siegel set* $\mathcal{S} = \mathcal{S}_{\omega,\tau}$ in X (relative to the $\mathbb{Q}$-Weyl chamber $\overline{A^+} \cdot x_0$) is a subset of X of the form $\mathcal{S}_{\omega,\tau} = \omega A_\tau \cdot x_0$ where ω is relatively compact in UM and, for $\tau > 0$, $A_\tau = \{a \in A \mid \alpha(a) \geq \tau\ ,\ \alpha \in \Delta\}$. If we define $a_0 \in A$ by $\alpha(a_0) = \tau$ for all $\alpha \in \Delta$, then $A_\tau = A_1 a_0 = \overline{A^+} a_0$ and $\mathcal{C} = A_\tau \cdot x_0 \subset \mathcal{S}$ is a (translate of a) $\mathbb{Q}$-Weyl chamber in X. Siegel sets provide the building blocks for (approximate) fundamental domains for arithmetic groups. A subset $\Omega \subset X$ is called a *fundamental set* for an arithmetic group Γ if the following two conditions hold

(i) $X = \Gamma \cdot \Omega$;

(ii) for every $q \in \mathbf{G}(\mathbb{Q})$ the set $\{\gamma \in \Gamma \mid q\Omega \cap \gamma\Omega \neq \varnothing\}$ is finite.

The existence of fundamental sets is guaranteed by reduction theory for arithmetic groups (see [B1] §13 and §15).

PROPOSITION 2.1 (Borel, Harish-Chandra). *Let* $\mathbf{G}$ *be a semisimple algebraic group defined over* $\mathbb{Q}$ *with associated Riemannian symmetric space* $X = G/K$. *Let* $\mathbf{P}$ *be a minimal parabolic* $\mathbb{Q}$*-subgroup of* $\mathbf{G}$ *and let* Γ *be an arithmetic subgroup of* $\mathbf{G}(\mathbb{Q})$. *Then there exists a generalized Siegel set* $\mathcal{S} = \mathcal{S}_{\omega,\tau}$ *(with respect to* $\overline{A^+} \cdot x_0$*) such that, for a (fixed) set* $\{q_i \mid 1 \leq i \leq m\}$ *of representatives of the* finite *set of double cosets* $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$*, the union* $\Omega = \bigcup_{i=1}^m q_i \cdot \mathcal{S}$ *is a fundamental set (of finite volume) for* Γ *in* X.

It will be useful in the sequel to dispose of geometric interpretations of the above algebraic concepts and assertions.

First recall that the symmetric space X, as a Riemannian manifold of nonpositive curvature, has an *ideal boundary at infinity* $\partial_\infty X$. The latter is defined as the set of equivalence classes of asymptotic geodesic rays (see [BGS]). In the same way one also defines the ideal boundary at infinity $\partial_\infty V$ of $V = \Gamma\backslash X$. If Γ is an arithmetic lattice in a group $\mathbf{G}$ of $\mathbb{Q}$-rank $q = 1$, the boundary $\partial_\infty V$ of the associated locally symmetric space consists of m points (corresponding to the cusps), where m is as in Proposition 2.1. For $\mathbb{Q}$-rank $q \geq 2$ it turns out that $\partial_\infty V$ is isomorphic to a finite simplicial complex $\Gamma\backslash|\mathcal{T}|$, a geometric realization of the Tits building of $\mathbf{G}$ modulo Γ (see [JM] and [L1]). We recall the construction of the latter.

Let $\mathcal{P}$ be the set of all parabolic $\mathbb{Q}$-subgroups of $\mathbf{G}$. The conjugacy classes of elements of $\mathcal{P}$ are in one-to-one correspondence with the subsets Θ of the set Δ of simple $\mathbb{Q}$-roots. Every conjugacy class has a standard representative denoted by $\mathbf{P}_\Theta$. One can show that the sets of double cosets $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}_\Theta(\mathbb{Q})$ are *finite* for all Θ (see [B1], §15.6). Let $\triangle = [e_1, \ldots, e_q] \subset \mathbb{R}^q$ denote a standard geometric $q-1$ simplex ($q = \mathbb{Q}$-rank of $\mathbf{G}$). If $\Delta = \{\alpha_1, \ldots, \alpha_q\}$ and $\Delta - \Theta = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ with $1 \leq i_1 < \ldots < i_s \leq q$, we define the boundary simplex $\triangle(\Theta)$ of $\triangle$ as $\triangle(\Theta) := [e_{i_1}, \ldots, e_{i_s}]$. Let $\mathbf{P}$ be a *minimal* parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ and let the set $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ be represented by $\{q_1, \ldots q_m\}$ (see Proposition 2.1). We take m copies $\triangle^j = [e^j_1, \ldots, e^j_q]$ of $\triangle$ with faces $\triangle^j(\Theta)$ corresponding to Θ. The corresponding homeomorphisms $\triangle \simeq \triangle^j$ are denoted by φ_j. The simplicial complex $\Gamma\backslash|\mathcal{T}|$, which provides a geometric realization of the quotient of the Tits building of $\mathbf{G}$ modulo Γ, is constructed from the simplices $\triangle^1, \ldots, \triangle^m$ through the following *incidence relations*:

Two simplices $\triangle^j$ and $\triangle^l$ are pasted together along the faces $\triangle^j(\Theta)$ and $\triangle^l(\Theta)$ by the homeomorphism $\varphi_j \circ \varphi_l^{-1}|_{\triangle^l(\Theta)}$ if and only if

$$\Gamma q_j \mathbf{P}_\Theta(\mathbb{Q}) = \Gamma q_l \mathbf{P}_\Theta(\mathbb{Q}).$$

We remark that the points of $\Gamma\backslash|\mathcal{T}|$ are in one-to-one correspondence with equivalence classes of geodesic rays in the locally symmetric space $V = \Gamma\backslash X$ (see [Hat], [L1] and [JM]).

2.2. An exhaustion by polyhedra

We index the "edges" of the Weyl chamber $\overline{\mathfrak{a}^+}$ (or equivalently of $\overline{A^+} \cdot x_0$) by *simple* $\mathbb{Q}$-roots. More precisely, the edges of $\overline{A^+} \cdot x_0$ are given by geodesic rays $c_\alpha(t) = \exp(tH_\alpha) \cdot x_0$ where $H_\alpha \in \overline{\mathfrak{a}^+}$, $\|H_\alpha\| = 1$ and $\beta(H_\alpha) = 0$ for $\beta \neq \alpha$ ($\alpha, \beta \in \Delta$). We further write $c_{k\alpha}$ for the edges $q_k a_0 c_\alpha$ of the chambers $q_k C$ in the fundamental set Ω (see Section 2.1 for the notation). If a geodesic ray c represents a point $z \in \partial_\infty X$ we write $z = c(\infty)$. The group G act naturally on $\partial_\infty X$ through $g \cdot c(\infty) = (g \cdot c)(\infty)$. For every $\alpha \in \Delta$ the isotropy group of $c_\alpha(\infty)$ under that action coincides with the (maximal) parabolic subgroups $P_{\Delta - \{\alpha\}}$ introduced above (see [L2] Lemma 1.2).

To a geodesic ray $c\colon [0, \infty) \longrightarrow X$ (parametrized by arc-length) which represents a point z in the ideal boundary $\partial_\infty X$ of X is associated a *Busemann function on X at z* given by

$$h_z\colon X \longrightarrow \mathbb{R} \quad ; \quad h_z(x) = \lim_{t\to\infty} \left[d\big(x, c(t)\big) - t\right].$$

The level sets of a Busemann function are *horospheres*, which foliate the symmetric space. We denote the Busemann functions which correspond to the rays $c_{k\alpha}$ by $h_{k\alpha}$. Note that $h_{k\alpha}\big(c_{k\alpha}(t)\big)$ tends to $-\infty$ if the arc-length t of the geodesic $c_{k\alpha}$ tends to $+\infty$.

In contrast to an exact fundamental domain there are not only points on the boundary of a fundamental set Ω but possibly also interior points which are identified under the action of Γ. However, there is only a finite set of isometries $\gamma \in \Gamma$ with $\gamma\Omega \cap \Omega \neq \varnothing$. Furthermore it suffices to look at the (finite) set $\mathcal{D}$ of those γ for which this intersection is not relatively compact in X (all other intersections are contained in some compact subset of Ω). It turns out that every $\gamma \in \mathcal{D}$ has the crucial property that there are indices i, j such that $q_j^{-1}\gamma q_i$ is parabolic i.e. fixes at least one point in the ideal boundary $\partial_\infty X$ (see [L2] Proposition 2.2). Then for every $\gamma \in \mathcal{D}$ there are indices i, j, α such the family of horospheres of the form $h_{i\alpha}^{-1}(s), s \in \mathbb{R}$, is mapped isometrically to the family $h_{j\alpha}^{-1}(s), s \in \mathbb{R}$ (see [L2] Lemma 3.2). These identifications correspond to the incidence relations described above in the construction of the simplicial complex $\Gamma\backslash|\mathcal{T}|$. (To see this one has to use the fact that the Siegel set at infinity $\partial_\infty(q_j\mathcal{S})$ is canonically isomorphic to $\triangle^j = [e_1^j, \ldots, e_q^j]$.) The main technical step is then to renormalize the Busemann functions as $\tilde{h}_{i\alpha} = h_{i\alpha} - s_{ij}$ (for certain constants s_{ij}) in such a way that each $\gamma \in \mathcal{D}$ maps a horosphere of some given level, say $\{\tilde{h}_{i\alpha} = s\}$, to another one, $\{\tilde{h}_{j\alpha} = s\}$, of the *same* level s (see [L2] Lemma 3.4). This fact finally allows us to truncate the constituents $q_i\mathcal{S}$ of the fundamental set Ω by removing the open horoballs $\mathcal{B}_{i\alpha}(s) := \{\tilde{h}_{i\alpha} < -\tau_\alpha s\}$ (for certain constants τ_α and for $s > 0$ sufficiently large). The above construction guarantees that the truncated fundamental set $\Omega(s) := \bigcup_{i=1}^m q_i\mathcal{S}(s)$ of Ω is relatively compact in X and invariant under the (restricted) action of Γ. Moreover for s sufficiently large the Γ-invariant "core" $X(s) := \Gamma \cdot \Omega(s)$ can be written as the complement in X of a union of (countably many) open horoballs: $X(s) = X - \Gamma \cdot \bigcup_{i=1}^m \bigcup_{\alpha \in \Delta} \mathcal{B}_{i\alpha}(s)$ (see [L3] Theorem 3.6). These horoballs are disjoint if and only if Γ is an arithmetic subgroup of a $\mathbb{Q}$-rank 1 group. The projection $\pi : X \longrightarrow V$ maps $X(s)$ to a compact submanifold with corners $V(s)$ of V whose fundamental group is isomorphic to Γ. The "centers" of the projected horoballs in $\partial_\infty V$ are in bijection with the vertices of $\Gamma\backslash|\mathcal{T}|$. The exhaustion function h is eventually defined in such a way that its level sets coincide with the boundaries $\partial V(s)$. We summarize the result in the following proposition (see [L2] Theorem 4.2).

PROPOSITION 2.2. *Let X be a Riemannian symmetric space of noncompact type and $\mathbb{R}$-rank ≥ 2 and let Γ be an irreducible, torsion-free, non-uniform lattice in the group of isometries of X. On the locally symmetric space $V = \Gamma\backslash X$ there exists a piecewise real analytic exhaustion function $h : V \longrightarrow [0, \infty)$ such that, for each $s \geq 0$, the sublevel set $V(s) := \{h \leq s\}$ is a Riemannian polyhedron in V. Moreover the level sets $\{h = s\} = \partial V(s)$ consist of projections of pieces of horospheres in X.*

Each polyhedron $V(s)$ is homotopically equivalent to V. More precisely we have

PROPOSITION 2.3. *For every sufficiently large s the locally symmetric space V is homeomorphic to the interior of the polyhedron $V(s)$ in V, and $V(s)$ is a strong deformation retract of V.*

For the proof see [L3], Theorems 5.2 and 5.5.

3. ESTIMATES FOR THE BOUNDARY SUBPOLYHEDRA

We wish to apply Proposition 1.1 to the polyhedra $V(s)$ in the above exhaustion and then take the limit for $s \to \infty$. To that end we need estimates for the second fundamental forms and the volumes of the (lower dimensional) boundary polyhedra.

For each Siegel set $\mathcal{S}_i := q_i \mathcal{S}$ which is part of the fundamental set Ω we have its truncated part

$$\mathcal{S}_i(s) := \mathcal{S}_i - \bigcup_{\alpha \in \Delta} \left(\mathcal{B}_{i\alpha}(s) \cap \mathcal{S}_i \right) .$$

The top dimensional boundary faces of $\mathcal{S}_i(s)$ in $\mathcal{S}_i$ (resp. of $\Omega(s)$ in Ω) are subsets of horospheres:

$$\mathcal{H}_{i\alpha}(s) := \{\tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s\} \cap \mathcal{S}_i(s) \ , \qquad \alpha \in \Delta \ .$$

The "horospherical" pieces $\mathcal{H}_{i\alpha}(s)$ together with their Γ-translates form the boundary of the manifold with corners $X(s)$ in X. For any nonempty subset Θ of Δ we set

$$\mathcal{H}_{i\Theta}(s) := \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \mathcal{S}_i(s) .$$

The various boundary subpolyhedra of $V(s)$ are then unions of projections of the pieces $\mathcal{H}_{i\Theta}(s)$ under the canonical projection $\pi : X \to V$. More precisely, as explained in Section 2, for any subset $\Theta \subset \Delta$, we have the equivalence relation on the set $I = \{1, \dots, m\}$

$$j \sim_\Theta l \quad \text{if and only if} \quad \Gamma q_j P_\Theta = \Gamma q_l P_\Theta$$

(the q_i are as in Proposition 2.1). This relation $\sim_\Theta$ induces a partition, $I(\Theta)$, of the set I whose components will be denoted by E. Let $n = \dim X = \dim V$, let k be the cardinality of Θ and let $E \in I(\Theta)$. Then $V_E^{n-k}(s) := \pi\left(\bigcup_{i \in E} \mathcal{H}_{i\Theta}(s)\right)$ is a $(n-k)$-dimensional boundary polyhedron of $V(s)$; and moreover, any boundary polyhedron arises in this way (see [L3] §4).