# 2.2. An exhaustion by polyhedra

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

standard geometric $q-1$ simplex ($q = \mathbb{Q}$-rank of $\mathbf{G}$). If $\Delta = \{\alpha_1, \dots, \alpha_q\}$ and $\Delta - \Theta = \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$ with $1 \leq i_1 < \dots < i_s \leq q$, we define the boundary simplex $\triangle(\Theta)$ of $\triangle$ as $\triangle(\Theta) := [e_{i_1}, \dots, e_{i_s}]$. Let $\mathbf{P}$ be a *minimal* parabolic $\mathbb{Q}$-subgroup of $\mathbf{G}$ and let the set $\Gamma\backslash\mathbf{G}(\mathbb{Q})/\mathbf{P}(\mathbb{Q})$ be represented by $\{q_1, \dots q_m\}$ (see Proposition 2.1). We take $m$ copies $\triangle^j = [e^j_1, \dots, e^j_q]$ of $\triangle$ with faces $\triangle^j(\Theta)$ corresponding to $\Theta$. The corresponding homeomorphisms $\triangle \simeq \triangle^j$ are denoted by $\varphi_j$. The simplicial complex $\Gamma\backslash|\mathcal{T}|$, which provides a geometric realization of the quotient of the Tits building of $\mathbf{G}$ modulo $\Gamma$, is constructed from the simplices $\triangle^1, \dots, \triangle^m$ through the following *incidence relations*:

Two simplices $\triangle^j$ and $\triangle^l$ are pasted together along the faces $\triangle^j(\Theta)$ and $\triangle^l(\Theta)$ by the homeomorphism $\varphi_j \circ \varphi_l^{-1}|_{\triangle^l(\Theta)}$ if and only if

$$\Gamma q_j \mathbf{P}_\Theta(\mathbb{Q}) = \Gamma q_l \mathbf{P}_\Theta(\mathbb{Q}).$$

We remark that the points of $\Gamma\backslash|\mathcal{T}|$ are in one-to-one correspondence with equivalence classes of geodesic rays in the locally symmetric space $V = \Gamma\backslash X$ (see [Hat], [L1] and [JM]).

## 2.2. An exhaustion by polyhedra

We index the "edges" of the Weyl chamber $\overline{\mathfrak{a}^+}$ (or equivalently of $\overline{A^+}\cdot x_0$) by *simple* $\mathbb{Q}$-roots. More precisely, the edges of $\overline{A^+}\cdot x_0$ are given by geodesic rays $c_\alpha(t) = \exp(tH_\alpha) \cdot x_0$ where $H_\alpha \in \overline{\mathfrak{a}^+}$, $\|H_\alpha\| = 1$ and $\beta(H_\alpha) = 0$ for $\beta \neq \alpha$ $(\alpha, \beta \in \Delta)$. We further write $c_{k\alpha}$ for the edges $q_k a_0 c_\alpha$ of the chambers $q_k C$ in the fundamental set $\Omega$ (see Section 2.1 for the notation). If a geodesic ray $c$ represents a point $z \in \partial_\infty X$ we write $z = c(\infty)$. The group $G$ act naturally on $\partial_\infty X$ through $g\cdot c(\infty) = (g\cdot c)(\infty)$. For every $\alpha \in \Delta$ the isotropy group of $c_\alpha(\infty)$ under that action coincides with the (maximal) parabolic subgroups $P_{\Delta-\{\alpha\}}$ introduced above (see [L2] Lemma 1.2).

To a geodesic ray $c\colon [0,\infty) \longrightarrow X$ (parametrized by arc-length) which represents a point $z$ in the ideal boundary $\partial_\infty X$ of $X$ is associated a *Busemann function on $X$ at $z$* given by

$$h_z\colon X \longrightarrow \mathbb{R} \quad ; \quad h_z(x) = \lim_{t\to\infty} \left[d\big(x, c(t)\big) - t\right].$$

The level sets of a Busemann function are *horospheres*, which foliate the symmetric space. We denote the Busemann functions which correspond to the rays $c_{k\alpha}$ by $h_{k\alpha}$. Note that $h_{k\alpha}\big(c_{k\alpha}(t)\big)$ tends to $-\infty$ if the arc-length $t$ of the geodesic $c_{k\alpha}$ tends to $+\infty$.

In contrast to an exact fundamental domain there are not only points on the boundary of a fundamental set $\Omega$ but possibly also interior points which are identified under the action of $\Gamma$. However, there is only a finite set of isometries $\gamma \in \Gamma$ with $\gamma\Omega \cap \Omega \neq \varnothing$. Furthermore it suffices to look at the (finite) set $\mathcal{D}$ of those $\gamma$ for which this intersection is not relatively compact in $X$ (all other intersections are contained in some compact subset of $\Omega$). It turns out that every $\gamma \in \mathcal{D}$ has the crucial property that there are indices $i,j$ such that $q_j^{-1}\gamma q_i$ is parabolic i.e. fixes at least one point in the ideal boundary $\partial_\infty X$ (see [L2] Proposition 2.2). Then for every $\gamma \in \mathcal{D}$ there are indices $i,j,\alpha$ such the family of horospheres of the form $h_{i\alpha}^{-1}(s), s \in \mathbb{R}$, is mapped isometrically to the family $h_{j\alpha}^{-1}(s), s \in \mathbb{R}$ (see [L2] Lemma 3.2). These identifications correspond to the incidence relations described above in the construction of the simplicial complex $\Gamma\backslash|\mathcal{T}|$. (To see this one has to use the fact that the Siegel set at infinity $\partial_\infty(q_j\mathcal{S})$ is canonically isomorphic to $\triangle^j = [e_1^j, \dots, e_q^j]$.) The main technical step is then to renormalize the Busemann functions as $\tilde{h}_{i\alpha} = h_{i\alpha} - s_{ij}$ (for certain constants $s_{ij}$) in such a way that each $\gamma \in \mathcal{D}$ maps a horosphere of some given level, say $\{\tilde{h}_{i\alpha} = s\}$, to another one, $\{\tilde{h}_{j\alpha} = s\}$, of the *same* level $s$ (see [L2] Lemma 3.4). This fact finally allows us to truncate the constituents $q_i\mathcal{S}$ of the fundamental set $\Omega$ by removing the open horoballs $\mathcal{B}_{i\alpha}(s) := \{\tilde{h}_{i\alpha} < -\tau_\alpha s\}$ (for certain constants $\tau_\alpha$ and for $s > 0$ sufficiently large). The above construction guarantees that the truncated fundamental set $\Omega(s) := \bigcup_{i=1}^m q_i\mathcal{S}(s)$ of $\Omega$ is relatively compact in $X$ and invariant under the (restricted) action of $\Gamma$. Moreover for $s$ sufficiently large the $\Gamma$-invariant "core" $X(s) := \Gamma \cdot \Omega(s)$ can be written as the complement in $X$ of a union of (countably many) open horoballs: $X(s) = X - \Gamma \cdot \bigcup_{i=1}^m \bigcup_{\alpha\in\Delta} \mathcal{B}_{i\alpha}(s)$ (see [L3] Theorem 3.6). These horoballs are disjoint if and only if $\Gamma$ is an arithmetic subgroup of a $\mathbb{Q}$-rank 1 group. The projection $\pi : X \longrightarrow V$ maps $X(s)$ to a compact submanifold with corners $V(s)$ of $V$ whose fundamental group is isomorphic to $\Gamma$. The "centers" of the projected horoballs in $\partial_\infty V$ are in bijection with the vertices of $\Gamma\backslash|\mathcal{T}|$. The exhaustion function $h$ is eventually defined in such a way that its level sets coincide with the boundaries $\partial V(s)$. We summarize the result in the following proposition (see [L2] Theorem 4.2).

PROPOSITION 2.2. *Let $X$ be a Riemannian symmetric space of noncompact type and $\mathbb{R}$-rank $\geq 2$ and let $\Gamma$ be an irreducible, torsion-free, non-uniform lattice in the group of isometries of $X$. On the locally symmetric space $V = \Gamma\backslash X$ there exists a piecewise real analytic exhaustion function $h : V \longrightarrow [0,\infty)$ such that, for each $s \geq 0$, the sublevel set $V(s) := \{h \leq s\}$ is a Riemannian polyhedron in $V$. Moreover the level sets $\{h = s\} = \partial V(s)$ consist of projections of pieces of horospheres in $X$.*

Each polyhedron $V(s)$ is homotopically equivalent to $V$. More precisely we have

PROPOSITION 2.3. *For every sufficiently large* $s$ *the locally symmetric space* $V$ *is homeomorphic to the interior of the polyhedron* $V(s)$ *in* $V$, *and* $V(s)$ *is a strong deformation retract of* $V$.

For the proof see [L3], Theorems 5.2 and 5.5.

## 3. ESTIMATES FOR THE BOUNDARY SUBPOLYHEDRA

We wish to apply Proposition 1.1 to the polyhedra $V(s)$ in the above exhaustion and then take the limit for $s \to \infty$. To that end we need estimates for the second fundamental forms and the volumes of the (lower dimensional) boundary polyhedra.

For each Siegel set $\mathcal{S}_i := q_i \mathcal{S}$ which is part of the fundamental set $\Omega$ we have its truncated part

$$\mathcal{S}_i(s) := \mathcal{S}_i - \bigcup_{\alpha \in \Delta} \left( \mathcal{B}_{i\alpha}(s) \cap \mathcal{S}_i \right).$$

The top dimensional boundary faces of $\mathcal{S}_i(s)$ in $\mathcal{S}_i$ (resp. of $\Omega(s)$ in $\Omega$) are subsets of horospheres :

$$\mathcal{H}_{i\alpha}(s) := \{ \tau_\alpha^{-1} \tilde{h}_{i\alpha} = -s \} \cap \mathcal{S}_i(s) \ , \qquad \alpha \in \Delta \ .$$

The "horospherical" pieces $\mathcal{H}_{i\alpha}(s)$ together with their $\Gamma$-translates form the boundary of the manifold with corners $X(s)$ in $X$. For any nonempty subset $\Theta$ of $\Delta$ we set

$$\mathcal{H}_{i\Theta}(s) := \bigcap_{\alpha \in \Theta} \mathcal{H}_{i\alpha}(s) \subset \mathcal{S}_i(s).$$

The various boundary subpolyhedra of $V(s)$ are then unions of projections of the pieces $\mathcal{H}_{i\Theta}(s)$ under the canonical projection $\pi : X \to V$. More precisely, as explained in Section 2, for any subset $\Theta \subset \Delta$, we have the equivalence relation on the set $I = \{1, \dots, m\}$

$$j \sim_\Theta l \quad \text{if and only if} \quad \Gamma q_j P_\Theta = \Gamma q_l P_\Theta$$

(the $q_i$ are as in Proposition 2.1). This relation $\sim_\Theta$ induces a partition, $I(\Theta)$, of the set $I$ whose components will be denoted by $E$. Let $n = \dim X = \dim V$, let $k$ be the cardinality of $\Theta$ and let $E \in I(\Theta)$. Then $V_E^{n-k}(s) := \pi\left(\bigcup_{i \in E} \mathcal{H}_{i\Theta}(s)\right)$ is a $(n-k)$-dimensional boundary polyhedron of $V(s)$; and moreover, any boundary polyhedron arises in this way (see [L3] §4).