

4. A NEW PROOF OF THE GAUSS-BONNET FORMULA

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **42 (1996)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$\pi_A : Y \rightarrow A$ we set $A_\Theta(s) := \pi_A \circ \mu^{-1}(\mathcal{H}_{1\Theta}(s)) \subset A$. The set $A_\Theta(s)$ is contained in an “affine” subspace of A of the form $a_1 a_*(s) A^{q-k}$ where $a_1 a_*(s) \in A$ and A^{q-k} is a $q-k$ -dimensional subgroup of A (see Sections 3 and 4 of [L2]). We denote the restriction of dv_A to A^{q-k} by $dv_{A^{q-k}}$; for $k = q$ we have $A^0 = e$ and we set $dv_{A^0} \equiv 1$. By Lemma 3.3 we have (for k equal to the number of elements of Θ)

$$\text{Vol}(V_E^{n-k}(s)) \prec \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}}.$$

Since the horospherical piece $\mathcal{H}_{1\Theta}(s)$ is part of a Siegel set $\mathcal{S}_{\omega,\tau}$ with ω relatively compact (and hence of finite volume) in UM we get

$$\begin{aligned} \int_{\mu^{-1}(\mathcal{H}_{1\Theta}(s))} \rho(a)^{-1} dv_U \wedge dv_Z \wedge dv_{A^{q-k}} &\prec \\ &\prec \int_{\omega} dv_U \wedge dv_Z \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}} \prec \int_{A_\Theta(s)} \rho(a)^{-1} dv_{A^{q-k}}. \end{aligned}$$

Also by definition of a Siegel set we have $\alpha(a) \geq \tau \succ 1$ for all $\alpha \in \Delta$. Moreover, the computations in the proof of Lemma 4.1 (and Lemma 3.5) in [L2] show that for all $\alpha \in \Theta$ one has $\alpha(a_1 a_*(s)) \succ e^{\mu_\alpha s}$ with $\mu_\alpha > 0$. Hence, as $\Theta \subset \Delta$ is not empty and since $\rho = \sum_{\alpha \in \Delta} c_\alpha \alpha$ ($c_\alpha > 0$), there is a uniform constant $c > 0$ such that $\rho(a)^{-1} \prec e^{-cs}$ for all $a \in A_\Theta(s)$. As noted above the set $A_\Theta(s)$ is contained in a $(q-k)$ -dimensional affine cone in A . It is similar (in the sense of Euclidean geometry) to $A_\Theta(0)$ with similarity factor s (see the proof of Lemma 4.1 in [L2]). Hence we eventually get $\int_{A_\Theta(s)} dv_{A^{q-k}} \prec s^{q-k}$ and the Lemma follows. \square

4. A NEW PROOF OF THE GAUSS-BONNET FORMULA

In this section we present a new simplified proof of the Gauss-Bonnet theorem for higher rank locally symmetric spaces.

THEOREM 4.1. *Let X be a Riemannian symmetric space of noncompact type and \mathbb{R} -rank ≥ 2 and let Γ be an irreducible, torsion-free (non-uniform) lattice in the group of isometries of X . Then for the locally symmetric space $V = \Gamma \backslash X$ the Gauss-Bonnet formula holds :*

$$\chi(V) = \int_V \Psi dv.$$

Proof. By Proposition 2.2 there is an exhaustion $V = \bigcup_{s \geq 0} V(s)$ of V by Riemannian polyhedra $V(s)$. Each polyhedron $V(s)$ in this exhaustion is equipped with the Riemannian metric induced by the one of V . Proposition 1.1 applied to $V(s)$ yields

$$\left| (-1)^n \chi'(V(s)) - \int_{V(s)} \Psi dv \right| \prec \sum_{k=1}^q \sum_E \int_{V_E^{n-k}(s)} \int_{O(p)} \|\Psi_{E,k}\| d\omega_{k-1} dv_E(p)$$

where $q = \dim A$ is the \mathbb{Q} -rank of \mathbf{G} (see Section 2.1) and where the index E runs through a finite set. As we remarked in Section 1 the function $\Psi_{E,k}$ is locally computable from the components of the metric and the curvature tensor of $V(s)$ and from the components of the second fundamental form of $V_E^{n-k}(s)$ in $V(s)$. The fact that V is locally symmetric together with Lemma 3.2 thus implies that $\|\Psi_{E,k}\| \prec 1$ for all E, k . Using Lemma 3.4 we conclude that

$$\left| (-1)^n \chi'(V(s)) - \int_{V(s)} \Psi dv \right| \prec \sum_{k,E} \text{Vol}(V_E^{n-k}(s)) \prec e^{-cs} \sum_{k=1}^q s^{q-k}.$$

By Proposition 2.3 we have $\chi'(V(s)) = \chi(V)$. The polyhedra $V(s)$ exhaust V and $\chi(V)$ is an integer; hence $(-1)^n \chi(V) = \int_{V(s)} \Psi dv$ for sufficiently large s . Finally, for n odd $\Psi \equiv 0$ by definition (see [AW]) and the claimed formula follows. \square

REFERENCES

- [AW] ALLENDOERFER, C. B. and A. WEIL. The Gauss-Bonnet theorem for Riemannian polyhedra. *Trans. Amer. Math. Soc.* 53 (1943), 101–129.
- [BGS] BALLMANN, W., M. GROMOV and V. SCHROEDER. *Manifolds of Nonpositive Curvature*. Boston, 1985.
- [B1] BOREL, A. *Introduction aux groupes arithmétiques*. Paris, 1969.
- [B2] —— Stable and real cohomology of arithmetic groups. *Ann. scient. Éc. Norm. Sup. 4^e série* 7 (1974), 235–272.
- [B3] —— *Linear Algebraic Groups*. Second edition, New York, 1991.
- [BS] BOREL, A. and J.-P. SERRE. Corners and arithmetic groups. *Comment. Math. Helv.* 48 (1973), 436–491.
- [BT] BOREL, A. and J. TITS. Groupes réductifs. *I.H.E.S. Publ. Math.* 27 (1965), 55–150.
- [C] CHERN, S. S. A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds. *Ann. of Math.* 45 (1944), 747–752.
- [CG1] CHEEGER, J. and M. GROMOV. On the characteristic numbers of complete manifolds of bounded curvature and finite volume. *Differential Geometry and Complex Analysis*. H. E. Rauch Memorial Volume (I. Chavel, H. M. Farkas, eds.), Berlin, 1985, 115–154.