§1. Introduction to F

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This paper is largely expository, and much of the material in it is standard. These notes originated from our interest in the question of whether or not F is amenable. They were expanded in order to make available Thompson's unpublished proofs (from [T1]) of the simplicity of T and V and Thurston's interpretations of F and T as the groups of orientation-preserving, piecewise integral projective homeomorphisms of the unit interval and the circle.

In §1 we define F as a group of piecewise linear homeomorphisms of the unit interval $[0,1]$, and then give some examples of elements of F. In §2 we represent elements of F as tree diagrams, and give a normal form for elements of F. Two standard presentations for F are given in §3. In §4 we prove several theorems about F; these are partly motivated by the question of whether F is an amenable group. In §5 we define T and give Thompson's proof that T is simple. In §6 we define V and give Thompson's proof that V is simple. In §7 we give W. Thurston's interpretations of F and T in terms of piecewise integral projective homeomorphisms.

The group that we are denoting F was originally denoted $\widehat{\mathbb{P}}$ in [T1] and $\mathfrak{P}'$ in [McT], and was denoted $\mathbb{P}$ in [T2]. It was denoted F in [BroG] in 1984, and it was also denoted F in [Bri], [BriS], [Bro1], [Bro3], [Fo], [FrH], [GhS], [Gre], [GreS], [GuS], and [St]. It is denoted G in [BieS].

The group that we are denoting T was originally denoted $\widehat{C}$ in [T1]. It was denoted T in [Bro1] in 1987 and was denoted T in [Bri] and [St]. However, it was denoted G in [GhS] and [Gre]. It is denoted S in [BieS].

The group that we are denoting V was originally denoted $\widehat{V}$ in [T1] and $\mathfrak{C}'$ in [McT], and was denoted $Ft({}^{\omega}2)$ in [T2]. It was denoted $G_{2,1}$ in [H] in 1974, and was denoted G in [Bro1], [Bro2], and [St].

We have not included here all of the known results about these groups, but we have included in the bibliography those references of which we are aware.

We thank the referee for supplying important references of which we were unaware and helping to clarify the exposition. We also thank Ross Geoghegan for helpful comments.

§1. Introduction to F

Let F be the set of piecewise linear homeomorphisms from the closed unit interval $[0,1]$ to itself that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2. Since derivatives are positive where they exist, elements of F

preserve orientation. Let $f \in F$, and let $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ be the points at which f is not differentiable. Then since $f(0) = 0$, $f(x) = a_1 x$ for $x_0 \le x \le x_1$, where a_1 is a power of 2. Likewise, since $f(x_1)$ is a dyadic rational number, $f(x) = a_2 x + b_2$ for $x_1 \le x \le x_2$, where a_2 is a power of 2 and b_2 is a dyadic rational number. It follows inductively that

$$f(x) = a_i x + b_i \quad \text{for} \quad x_{i-1} \le x \le x_i$$

and $i = 1, \ldots, n$, where a_i is a power of 2 and b_i is a dyadic rational number. It easily follows that $f^{-1} \in F$ and that f maps the set of dyadic rational numbers bijectively to itself. From this it is easy to see that F is closed under composition of functions. Thus F is a subgroup of the group of all homeomorphisms from $[0, 1]$ to $[0, 1]$. This group F is Thompson's group F.

EXAMPLE 1.1. Two functions in F are the functions A and B defined below.

$$A(x) = \begin{cases} \frac{x}{2}, & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ 2x - 1, & \frac{3}{4} \le x \le 1 \end{cases} \qquad B(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{4} \\ x - \frac{1}{8}, & \frac{3}{4} \le x \le \frac{7}{8} \\ 2x - 1, & \frac{7}{8} \le x \le 1 \end{cases}$$

A useful notation for functions f in F will be described next. Construct a rectangle with a top, which is viewed as the domain of f, and a bottom, which is viewed as the range of f. For every point x on the top where f is not differentiable, construct a line segment from x to $f(x)$ on the bottom. Call the result the *rectangle diagram* of f. By juxtaposing the rectangle diagrams of a pair of functions, it is easy to compute the rectangle diagram of their composition. We learned about rectangle diagrams from W. Thurston in 1975; they also appear in [BieS].

EXAMPLE 1.2. Figure 1 gives some examples of functions in F and their rectangle diagrams.

Now define functions $X_0, X_1, X_2, \ldots$ in F so that $X_0 = A$ and $X_n = A^{-(n-1)} B A^{n-1}$ for $n \ge 1$. From Example 1.2 it is easy to see that the rectangle diagram of X_n is as in Figure 2.

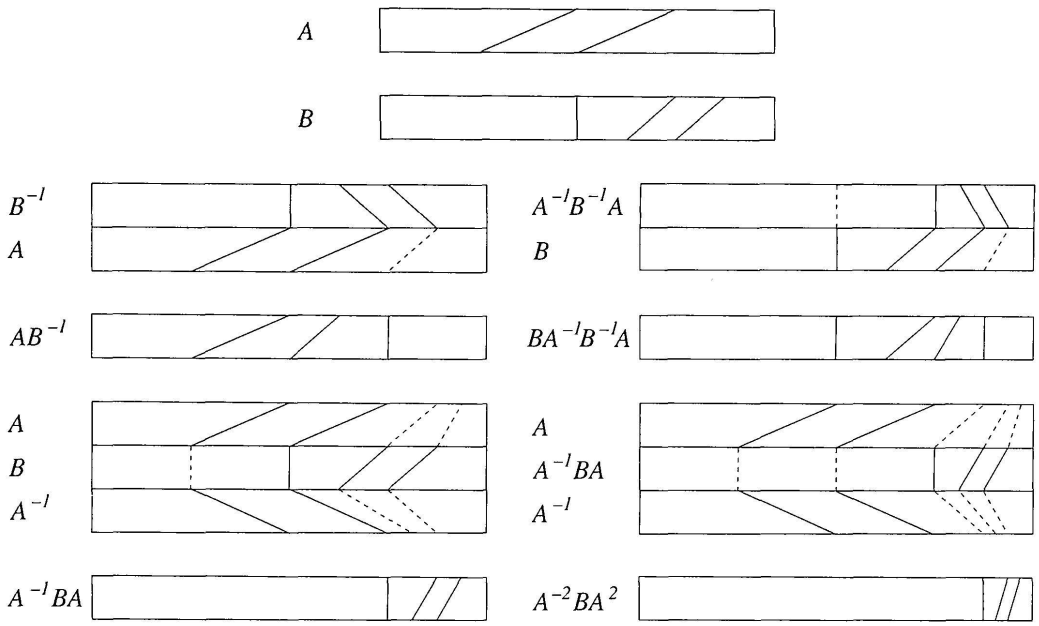

FIGURE 1
The rectangle diagrams of some elements of F

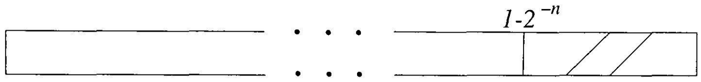

FIGURE 2
The rectangle diagram of X_n

§2. TREE DIAGRAMS

The notion of tree diagram is developed in this section. Tree diagrams are useful for describing functions in F; we first encountered them in [Bro1].

Define an *ordered rooted binary tree* to be a tree S such that i) S has a root v_0, ii) if S consists of more than v_0, then v_0 has valence 2, and iii) if v is a vertex in S with valence greater than 1, then there are exactly two edges $e_{v,L}$, $e_{v,R}$ which contain v and are not contained in the geodesic from v_0 to v. The edge $e_{v,L}$ is called a *left edge* of S, and $e_{v,R}$ is called a *right edge* of S. Vertices with valence 0 (in case of the trivial tree) or 1 in S will be called *leaves* of S. There is a canonical left-to-right linear ordering on the leaves of S. The *right side* of S is the maximal arc of right edges in S which begins at the root of S. The *left side* of S is defined analogously.

An *isomorphism* of ordered rooted binary trees is an isomorphism of rooted trees which takes left edges to left edges and right edges to right edges. An *ordered rooted binary subtree* S' of an ordered rooted binary tree S is an