§2. Tree diagrams

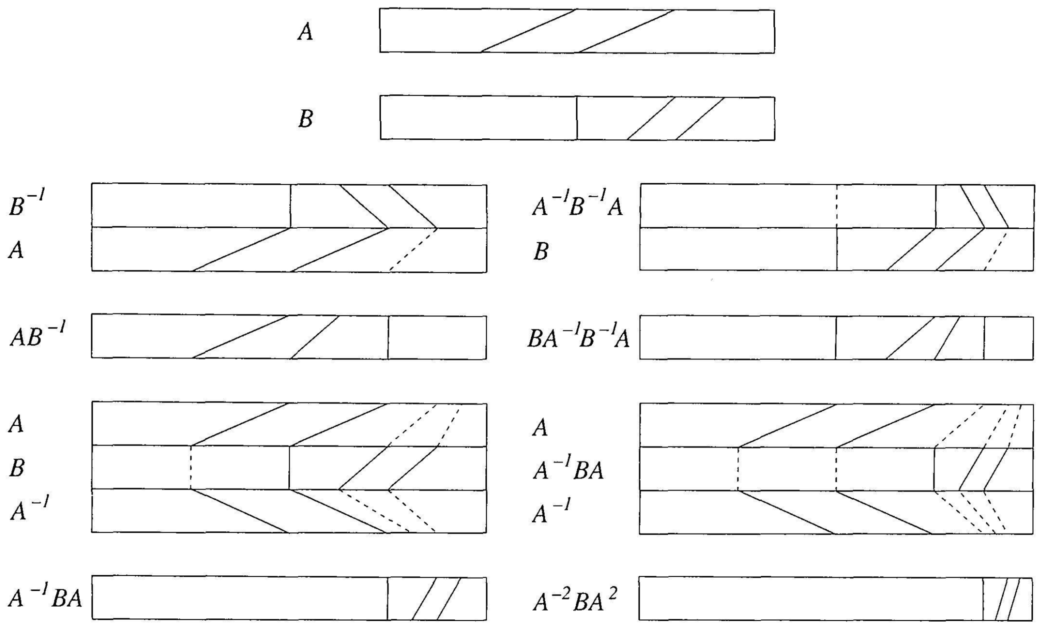

FIGURE 1
The rectangle diagrams of some elements of F

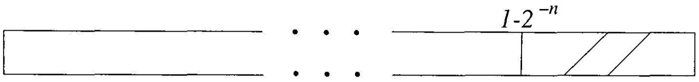

FIGURE 2
The rectangle diagram of X_n

§2. TREE DIAGRAMS

The notion of tree diagram is developed in this section. Tree diagrams are useful for describing functions in F; we first encountered them in [Bro1].

Define an *ordered rooted binary tree* to be a tree S such that i) S has a root v_0, ii) if S consists of more than v_0, then v_0 has valence 2, and iii) if v is a vertex in S with valence greater than 1, then there are exactly two edges $e_{v,L}$, $e_{v,R}$ which contain v and are not contained in the geodesic from v_0 to v. The edge $e_{v,L}$ is called a *left edge* of S, and $e_{v,R}$ is called a *right edge* of S. Vertices with valence 0 (in case of the trivial tree) or 1 in S will be called *leaves* of S. There is a canonical left-to-right linear ordering on the leaves of S. The *right side* of S is the maximal arc of right edges in S which begins at the root of S. The *left side* of S is defined analogously.

An *isomorphism* of ordered rooted binary trees is an isomorphism of rooted trees which takes left edges to left edges and right edges to right edges. An *ordered rooted binary subtree* S' of an ordered rooted binary tree S is an ordered rooted binary tree which is a subtree of S whose left edges are left edges of S, whose right edges are right edges of S, but whose root need not be the root of S.

EXAMPLE 2.1. The right side of the ordered rooted binary tree in Figure 3 is highlighted. Its leaves are labeled $0, \ldots, 5$ in order.

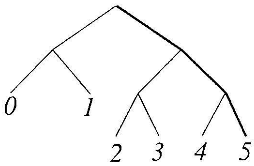

FIGURE 3
An ordered rooted binary tree with 6 leaves

Define a *standard dyadic interval* in $[0,1]$ to be an interval of the form $\left[\frac{a}{2^n}, \frac{a+1}{2^n}\right]$, where a, n are nonnegative integers with $a \leq 2^n - 1$.

There is a *tree of standard dyadic intervals*, $\mathcal{T}$, which is defined as follows. The vertices of $\mathcal{T}$ are the standard dyadic intervals in $[0,1]$. An edge of $\mathcal{T}$ is a pair (I, J) of standard dyadic intervals I and J such that either I is the left half of J, in which case (I, J) is a left edge, or I is the right half of J, in which case (I, J) is a right edge. It is easy to see that $\mathcal{T}$ is an ordered rooted binary tree. The tree of standard dyadic intervals is shown in Figure 4.

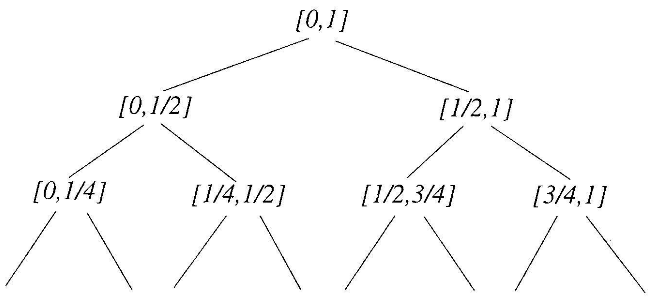

FIGURE 4
The tree $\mathcal{T}$ of standard dyadic intervals

Define a $\mathcal{T}$*-tree* to be a finite ordered rooted binary subtree of $\mathcal{T}$ with root [0,1]. Call the $\mathcal{T}$-tree with just one vertex the *trivial* $\mathcal{T}$-tree. For every nonnegative integer n, let $\mathcal{T}_n$ be the $\mathcal{T}$-tree with $n+1$ leaves whose right side has length n. $\mathcal{T}_3$ is shown in Figure 5.

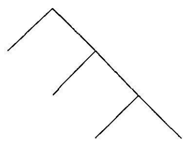

FIGURE 5
The $\mathcal{T}$-tree $\mathcal{T}_3$

Define a *caret* to be an ordered rooted binary subtree of $\mathcal{T}$ with exactly two edges. Every caret has the form of the rooted tree in Figure 6.

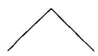

FIGURE 6
A caret

A partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ of $[0, 1]$ determines intervals $[x_{i-1}, x_i]$ for $i = 1, \ldots, n$ which are called the *intervals of the partition.* A partition of [0,1] is called a *standard dyadic partition* if and only if the intervals of the partition are standard dyadic intervals.

It is easy to see that the leaves of a $\mathcal{T}$-tree are the intervals of a standard dyadic partition. Conversely, the intervals of a standard dyadic partition determine finitely many vertices of $\mathcal{T}$, and it is easy to see that these vertices are the leaves of their convex hull, which is a $\mathcal{T}$-tree. Thus there is a canonical bijection between standard dyadic partitions and $\mathcal{T}$-trees.

LEMMA 2.2. *Let $f \in F$. Then there exists a standard dyadic partition $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$ such that f is linear on every interval of the partition and $0 = f(x_0) < f(x_1) < f(x_2) < \cdots < f(x_n) = 1$ is a standard dyadic partition.*

Proof. Choose a partition P of $[0, 1]$ whose partition points are dyadic rational numbers such that f is linear on every interval of P. Let $[a, b]$ be an interval of P. Suppose that the derivative of f on $[a, b]$ is 2^{-k}. Let m be an integer such that $m \geq 0$, $m + k \geq 0$, $2^m a \in \mathbf{Z}$, $2^m b \in \mathbf{Z}$, $2^{m+k} f(a) \in \mathbf{Z}$, and $2^{m+k} f(b) \in \mathbf{Z}$. Then $a < a + \frac{1}{2^m} < a + \frac{2}{2^m} < a + \frac{3}{2^m} < \cdots < b$ partitions $[a, b]$ into standard dyadic intervals, and $f(a) < f(a) + \frac{1}{2^{m+k}} < f(a) + \frac{2}{2^{m+k}} < f(a) + \frac{3}{2^{m+k}} < \cdots < f(b)$ partitions $\left[f(a), f(b)\right]$ into standard dyadic intervals. This easily proves Lemma 2.2. $\square$

Formally, a *tree diagram* is an ordered pair (R, S) of $\mathcal{T}$-trees such that R and S have the same number of leaves. This is rendered diagrammatically as follows:

$$R \to S.$$

The tree R is called the *domain tree* of the diagram, and S is called the *range tree* of the diagram.

Suppose given $f \in F$. Lemma 2.2 shows that there exist standard dyadic partitions P and Q such that f is linear on the intervals of P and maps them to the intervals of Q. To f is associated the tree diagram (R, S), where R is the $\mathcal{T}$-tree corresponding to P and S is the $\mathcal{T}$-tree corresponding to Q.

Because P and Q are not unique, there are many tree diagrams associated to f. Given one tree diagram (R, S) for f, another can be constructed by adjoining carets to R and S as follows. Let I be the n^{th} leaf of R for some positive integer n, and let J be the n^{th} leaf of S. Let I_1, I_2 be the leaves in order of the caret C with root I, and let J_1, J_2 be the leaves in order of the caret D with root J. Because f is linear on I and $f(I) = J$, it follows that $f(I_1) = J_1$ and $f(I_2) = J_2$. Thus (R', S') is a tree diagram for f, where $R' = R \cup C$ and $S' = S \cup D$.

In the other direction, if there exists a positive integer n such that the n^{th} and $(n+1)^{\text{th}}$ leaves of R, respectively S, are the vertices of a caret C, respectively D, then deleting all of C and D but the roots from R and S leads to a new tree diagram for f. If there do not exist such carets C, D in R, S, then the tree diagram (R, S) is said to be *reduced.*

In this paragraph it will be shown that there is exactly one reduced tree diagram for f. Suppose that (R, S) is a reduced tree diagram for f. It is easy to see that if I is a standard dyadic interval which is either a leaf of R or not in R, then $f(I)$ is a standard dyadic interval and f is linear on I. Conversely, if I is a standard dyadic interval such that $f(I)$ is a standard dyadic interval and f is linear on I, then I is either a leaf of R or not in R because (R, S) is reduced. Thus R is the unique $\mathcal{T}$-tree such that a standard dyadic interval I is either a leaf of R or not in R if and only if $f(I)$ is a standard dyadic interval and f is linear on I. This gives uniqueness of reduced tree diagrams.

Furthermore, if (R, S) is a tree diagram, then it is clear that there exists $f \in F$ such that f is linear on every leaf of R and f maps the leaves of R to the leaves of S.

Thus there is a canonical bijection between F and the set of reduced tree diagrams.

EXAMPLE 2.3. Figure 7 shows the reduced tree diagrams for A and B.

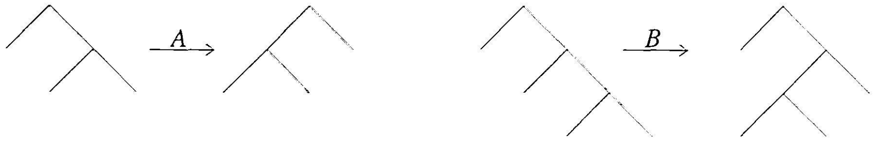

FIGURE 7
The reduced tree diagrams for A and B

From Figure 2 it is not difficult to see that, for $n \geq 0$, the reduced tree diagram for X_n is the tree diagram in Figure 8.

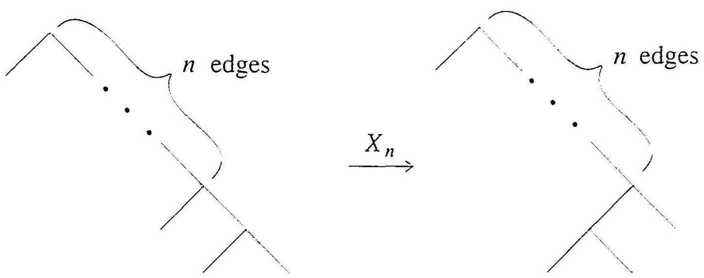

FIGURE 8
The reduced tree diagram for X_n

It is easy to see that if (Q, R) is a tree diagram for a function f in F and (R, S) is a tree diagram for a function g in F, then (Q, S) is a tree diagram for gf.

The following definition prepares for Theorem 2.5, which makes the correspondence between functions in F and tree diagrams more precise. Define the *exponents* of a $\mathcal{T}$-tree S as follows. Let $I_0, \ldots, I_n$ be the leaves of S in order. For every integer k with $0 \leq k \leq n$ let a_k be the length of the maximal arc of left edges in S which begins at I_k and which does not reach the right side of S. Then a_k is the k^{th} *exponent* of S.

EXAMPLE 2.4. Let S be the $\mathcal{T}$-tree shown in Figure 9.

The leaves of S are labeled $0, \ldots, 9$ in order, and the exponents of S in order are 2, 1, 0, 0, 1, 2, 0, 0, 0, 0.

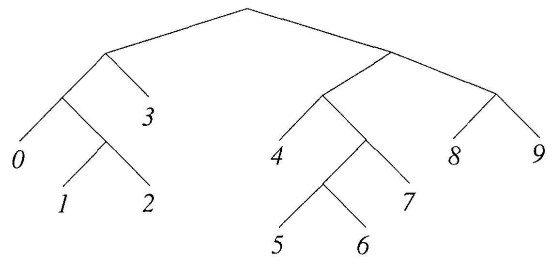

FIGURE 9

The $\mathcal{T}$-tree S

THEOREM 2.5. *Let R, S be $\mathcal{T}$-trees with $n+1$ leaves for some nonnegative integer n. Let $a_0, \ldots, a_n$ be the exponents of R, and let $b_0, \ldots, b_n$ be the exponents of S. Then the function in F with tree diagram (R, S) is $X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$. The tree diagram (R, S) is reduced if and only if i) if the last two leaves of R lie in a caret, then the last two leaves of S do not lie in a caret and ii) for every integer k with $0 \leq k < n$, if $a_k > 0$ and $b_k > 0$ then either $a_{k+1} > 0$ or $b_{k+1} > 0$.*

Proof. To prove the first statement of the theorem, by composing functions it suffices to prove that the function with tree diagram $(R, \mathcal{T}_n)$ is $X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$.

The proof of this will proceed by induction on $a = \sum_{i=0}^{n} a_i$. If $a = 0$, then $R = \mathcal{T}_n$, and the result is clear. Now suppose that $a > 0$ and that the result is true for smaller values of a. Let m be the smallest index such that $a_m > 0$. Then there are ordered rooted binary subtrees R_1, R_2, R_3 of R such that R has the form of the tree at the left of Figure 10.

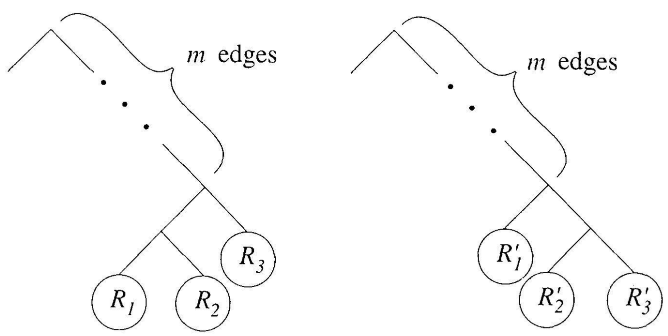

FIGURE 10

The $\mathcal{T}$-trees R and R'

Let R' be the $\mathcal{T}$-tree shown at the right of Figure 10, where R'_1, R'_2, R'_3 are isomorphic with R_1, R_2, R_3 as ordered rooted binary trees. According to Example 2.3, the function with tree diagram (R, R') is X_m^{-1}. If $a'_0, \ldots, a'_n$ are the exponents of R', then $a'_m = a_m - 1$ and $a'_k = a_k$ if $k \neq m$. Thus the induction hypothesis applies to R', and so the function with tree diagram $(R', \mathcal{T}_n)$ is $X_n^{-a'_n} \cdots X_2^{-a'_2} X_1^{-a'_1} X_0^{-a'_0}$. Again by composing functions, it follows that the function with tree diagram $(R, \mathcal{T}_n)$ is $X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0}$, as desired.

The second statement of the theorem is now easy to prove.

This proves Theorem 2.5. □

COROLLARY 2.6. *Thompson's group F is generated by A and B.*

COROLLARY-DEFINITION 2.7. *Every nontrivial element of F can be expressed in unique normal form*

$$X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n} X_n^{-a_n} \cdots X_2^{-a_2} X_1^{-a_1} X_0^{-a_0},$$

where $n, a_0, \ldots, a_n, b_0, \ldots, b_n$ are nonnegative integers such that i) exactly one of a_n and b_n is nonzero and ii) if $a_k > 0$ and $b_k > 0$ for some integer k with $0 \le k < n$, then $a_{k+1} > 0$ or $b_{k+1} > 0$. Furthermore, every such normal form function in F is nontrivial.

The functions in F of the form $X_0^{b_0} X_1^{b_1} X_2^{b_2} \cdots X_n^{b_n}$ with $b_k \ge 0$ for $k = 0, \ldots, n$ will be called *positive*. The positive elements of F are exactly those with tree diagrams having domain tree $\mathcal{T}_n$ for some nonnegative integer n. Inverses of positive elements will be called *negative*.

LEMMA 2.8. *The set of positive elements of F is closed under multiplication.*

Proof. Let f and g be positive elements of F. Let $(\mathcal{T}_m, R)$, respectively $(\mathcal{T}_n, S)$, be tree diagrams for f, respectively g. If the right side of S has length k, then it is easy to see that fg has a tree diagram with domain tree $\mathcal{T}_{n+\max\{m-k,0\}}$. Thus fg is positive. This proves Lemma 2.8. □

Fordham [Fo] gives a linear-time algorithm that takes as input the reduced tree diagram representing an element of Thompson's group F and gives as output the minimal length of a word in generators A and B representing that element. The algorithm can be modified to actually construct one, or all, minimal representatives. Fordham assigns a type to each caret of the tree pair; the minimal length is a simple function of the type sequences of the two trees.