§7. Piecewise integral projective structures

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THEOREM 6.9. *V_1 is simple.*

Proof. Suppose N is a nontrivial normal subgroup of V_1, and let $\theta\colon V_1 \to V_1/N$ be the quotient homomorphism. Then there is an element $g \in V_1$ with $g \neq 1$ and $\theta(g) = 1$. By Lemmas 5.6.iii), 5.6.iv), 6.7, 6.8.i) and Theorem 5.7 we have $g = p\pi C_n^m q^{-1}$ for some positive elements p and q, some integers m, n with $0 \leq m < n+2$, and some element $\pi \in \Pi(n)$. Then $\theta(\pi C_n^m) = \theta(p^{-1}q)$. Lemma 6.8.ii) implies that πC_n^m has finite order, say, k. Furthermore the subgroup of V_1 generated by A and B is torsion-free because it maps injectively to $F \subseteq V$ by Theorem 3.4. Hence either $(p^{-1}q)^k \neq 1$ and $\theta\big((p^{-1}q)^k\big) = 1$ or $\pi C_n^m \neq 1$ and $\theta(\pi C_n^m) = 1$. Suppose that $\pi C_n^m \neq 1$ and $\theta(\pi C_n^m) = 1$. If $m = 0$, then $\pi \neq 1$ and $\theta(\pi) = 1$. This implies that $\theta(\pi_0) = \theta(\pi_1)$, and hence by Lemma 6.5 that $\theta(\pi_0 C_2) = \theta(C_2\pi_1) = \theta(C_2\pi_0) = \theta(\pi_0\pi_1 C_2^2)$. But then $\theta(\pi_1 C_2) = 1$, so we may assume that $m > 0$. Next suppose that $m > 0$. Then $\pi C_n^m = \pi X_{n+1-m} C_{n+1}^m$ by Lemma 5.6.iii). Lemma 6.4 implies that there exists a nonnegative integer i and $\pi' \in \Pi(n+1)$ such that $\pi C_n^m = X_i \pi' C_{n+1}^m$. Thus we are in the above case in which $(p^{-1}q)^k \neq 1$ and $\theta\big((p^{-1}q)^k\big) = 1$.

In each case there is an element $h \in V_1$ such that $h \neq 1$, $\theta(h) = 1$, and h can be represented as a word in $A^{\pm 1}$, $B^{\pm 1}$, and $C^{\pm 1}$. Let $\alpha\colon T_1 \to V_1/N$ be the homomorphism defined by $\alpha(A) = \theta(A)$, $\alpha(B) = \theta(B)$, and $\alpha(C) = \theta(C)$. Then there is an element $h' \in T_1$ with $h' \neq 1$ and $\alpha(h') = 1$. Since T_1 is simple by Theorem 5.8, $\theta(A) = \theta(B) = \theta(C) = 1$. Because π_i and π_j are conjugate via a power of A, $\theta(\pi_i) = \theta(\pi_j)$ for all nonnegative integers i and j. By Lemma 6.6.ii) with $k = 1$, $m = 2$ and $n = 2$, $\theta(\pi_1) = \theta(C_2^2\pi_1) = \theta(\pi_0\pi_1 C_2^3) = \theta(\pi_0\pi_1)$, and hence $\theta(\pi_0) = 1$. This implies that the quotient group is trivial. $\square$

§7. PIECEWISE INTEGRAL PROJECTIVE STRUCTURES

The definition of piecewise integral projective structures is due to W. Thurston. These structures arise naturally on the boundaries of Teichmüller spaces of surfaces. The interpretations of F and T as groups of piecewise integral projective homeomorphisms are also due to Thurston; we learned this from him in 1975. Greenberg [Gr] used this interpretation in his study of these groups.

Fix a positive integer n.

The symbol $\triangle_n$ denotes the n-simplex $\{(x_1,\dots,x_{n+1}) \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i\}$. The n-simplex $\triangle_n$ is an orientable n-manifold with boundary. A *rational point* of $\triangle_n$ is a point $(x_1,\dots,x_{n+1}) \in \triangle_n$ with each $x_i \in \mathbf{Q}$.

Set $\mathbf{R}_+^{n+1} = \{(x_1,\dots,x_{n+1}) \in \mathbf{R}^{n+1} : x_i \geq 0 \text{ for } i = 1,\dots,n+1\}$. One defines $\rho : \mathbf{R}_+^{n+1} \setminus \{0\} \to \triangle_n$ using the projective structure of $\mathbf{R}^{n+1}$; that is, $\rho(x) = \frac{x}{|x|}$, where $|x| = \sum_{i=1}^{n+1} |x_i|$. Let $U \subset \triangle_n$. A function $f\colon U \to \triangle_n$ is *integral projective* if there exists $A \in GL(n+1,\mathbf{Z})$ such that $U \subset \{x \in \triangle_n : A(x) \in \mathbf{R}_+^{n+1}\}$ and $f = \rho \circ A\big|_U$. It is easily seen that an integral projective map is a homeomorphism onto its image.

A *rational subsimplex* of $\triangle_n$ is a subsimplex of $\triangle_n$ in which each vertex is a rational point; a *rational subdivision* of $\triangle_n$ is a simplicial subdivision in which each n-simplex is a rational subsimplex. An *integral subsimplex* of $\triangle_n$ is a subsimplex of $\triangle_n$ which is homeomorphic to $\triangle_n$ by an integral projective map. Similarly, an *integral subdivision* of $\triangle_n$ is a simplicial subdivision of $\triangle_n$ in which each n-simplex is an integral subsimplex of $\triangle_n$.

A *piecewise integral projective (PIP)* homeomorphism of $\triangle_n$ is a homeomorphism $f\colon \triangle_n \to \triangle_n$ such that there is an integral subdivision $\mathcal{S}$ of $\triangle_n$ with $f\big|_\sigma$ integral projective for each simplex σ of $\mathcal{S}$. Define $PIP(\triangle_n)$ to be the set of all PIP homeomorphisms of $\triangle_n$. We wish to prove that $PIP(\triangle_n)$ is a group by proving that it is closed under inversion and composition. It is easy to see that $PIP(\triangle_n)$ is closed under inversion. It is not immediately obvious that the composition of two PIP homeomorphisms is a PIP homeomorphism. The stumbling block is whether two integral subdivisions of $\triangle_n$ have a common refinement which is an integral subdivision. According to Exercise 5 on page 15 of [RS] their intersection is a cell complex which is a common refinement of both, and it is easy to see that the cells of this intersection complex have rational points as vertices. Proposition 2.9 of [RS] states that such a cell complex can be subdivided to a simplicial complex without introducing any new vertices. Hence to prove that $PIP(\triangle_n)$ is a group it suffices to prove the following theorem.

THEOREM 7.1. *Every rational subdivision of $\triangle_n$ has a refinement that is an integral subdivision.*

Proof. We define the *lift* of a rational point x in $\triangle_n$ to be the unique point $\widetilde{x}$ in $\mathbf{Z}^{n+1} \cap \mathbf{R}_+^{n+1}$ such that $\rho(\widetilde{x}) = x$ and the greatest common divisor of the coordinates of $\widetilde{x}$ is 1. We define the *index* of an n-dimensional rational subsimplex σ of $\triangle_n$ as follows. Let $v_1, \dots, v_{n+1}$ be the vertices of σ. Then the subgroup of $\mathbf{Z}^{n+1}$ generated by $\widetilde{v}_1, \dots, \widetilde{v}_{n+1}$ has finite index in $\mathbf{Z}^{n+1}$. The index $\mathrm{ind}(\sigma)$ of σ is by definition this index. Equivalently, $\mathrm{ind}(\sigma) = |\det(\widetilde{v}_1, \dots, \widetilde{v}_{n+1})|$, the absolute value of the determinant of the matrix whose columns are $\widetilde{v}_1, \dots, \widetilde{v}_{n+1}$. It is easy to see that $\mathrm{ind}(\sigma) = 1$ if and only if σ is integral.

The argument will proceed as follows. Let $\mathcal{S}$ be a rational subdivision of $\triangle_n$. Suppose that σ is an n-simplex in $\mathcal{S}$ with $\mathrm{ind}(\sigma) > 1$. A rational point v in σ will be suitably chosen. We will let $\mathcal{R}$ be the simplicial complex obtained from $\mathcal{S}$ by starring at v as on page 15 of [RS]. If τ is an n-simplex in $\mathcal{R}$ which does not contain v, then $\tau \in \mathcal{S}$. If τ is an n-simplex in $\mathcal{R}$ which contains v, then we will prove that $\mathrm{ind}(\tau)$ is less than the index of the n-simplex in $\mathcal{S}$ which contains τ. From this it easily follows that performing finitely many such starring subdivisions yields a rational subdivision of $\triangle_n$ all of whose n-simplices have index 1, and so this subdivision is integral, as desired.

So let $\mathcal{S}$ be a rational subdivision of $\triangle_n$, and let σ be an n-simplex in $\mathcal{S}$ with $\mathrm{ind}(\sigma) > 1$. Let the vertices of σ be $v_1, \dots, v_{n+1}$. Since $\mathrm{ind}(\sigma) > 1$, there exists $u \in \mathbf{Z}^{n+1}$ and an integer $m > 1$ such that mu lies in the subgroup of $\mathbf{Z}^{n+1}$ generated by $\widetilde{v}_1, \dots, \widetilde{v}_{n+1}$ but u does not. Let $a_1, \dots, a_{n+1}$ be integers such that $mu = \sum_{i=1}^{n+1} a_i \widetilde{v}_i$. For every integer i with $1 \le i \le n+1$ let b_i be an integer such that $0 \le a_i + mb_i < m$. Then

$$m\left(u + \sum_{i=1}^{n+1} b_i \widetilde{v}_i\right) = \sum_{i=1}^{n+1} (a_i + mb_i)\widetilde{v}_i .$$

Because u is not in the subgroup of $\mathbf{Z}^{n+1}$ generated by $\widetilde{v}_1, \dots, \widetilde{v}_{n+1}$, it is impossible that $a_i + mb_i = 0$ for $i = 1, \dots, n+1$. Reindex if necessary so that $a_i + mb_i \neq 0$ if $i \le k$ and $a_i + mb_i = 0$ if $i > k$ for some integer k with $1 \le k \le n+1$. The vector $w = u + \sum_{i=1}^{n+1} b_i \widetilde{v}_i$ is a positive rational linear combination of $\widetilde{v}_1, \dots, \widetilde{v}_k$, and so $v = \rho(w)$ is a rational point of $\triangle_n$ which lies in the open simplex with vertices $v_1, \dots, v_k$. Since $w \in \mathbf{Z}^{n+1} \cap \mathbf{R}_+^{n+1}$, w is a positive integer mulitple of $\widetilde{v}$. It follows that $\widetilde{v} = \sum_{j=1}^{k} c_j \widetilde{v}_j$ for rational numbers $c_1, \dots, c_k$ with $0 < c_j < 1$.

Now let $\mathcal{R}$ be the simplicial complex obtained from $\mathcal{S}$ by starring at v. Let τ be an n-simplex in $\mathcal{R}$ which contains v. Let σ' be the n-simplex in $\mathcal{S}$ which contains τ. Then $v_1, \dots, v_k$ are vertices of σ', and so the vertices of σ' have the form $v_1, \dots, v_k, v'_{k+1}, \dots, v'_{n+1}$. Hence the vertices of τ have the form $v_1, \dots, \widehat{v}_i, \dots v_k, v'_{k+1}, \dots, v'_{n+1}, v$ for some $i \in \{1, \dots, k\}$. Thus

$$\begin{aligned}\operatorname{ind}(\tau) &= \left|\det(\widetilde{v}_1, \dots, \widehat{\widetilde{v}_i}, \dots, \widetilde{v}_k, \widetilde{v}'_{k+1}, \dots, \widetilde{v}'_{n+1}, \widetilde{v})\right| \\ &= \left|\det(\widetilde{v}_1, \dots, \widehat{\widetilde{v}_i}, \dots, \widetilde{v}_k, \widetilde{v}'_{k+1}, \dots, \widetilde{v}'_{n+1}, \sum_{j=1}^{k} c_j \widetilde{v}_j)\right| \\ &= \left|\sum_{j=1}^{k} c_j \det(\widetilde{v}_1, \dots, \widehat{\widetilde{v}_i}, \dots, \widetilde{v}_k, \widetilde{v}'_{k+1}, \dots, \widetilde{v}'_{n+1}, \widetilde{v}_j)\right|.\end{aligned}$$

In the last expression we have a linear combination of k determinants of which all but one are 0 because the corresponding matrices have two equal columns. Hence $\operatorname{ind}(\tau) = c_i \operatorname{ind}(\sigma') < \operatorname{ind}(\sigma')$. This completes the proof of Theorem 7.1. $\square$

We denote by $PIP^+(\triangle_n)$ the subset of $PIP(\triangle_n)$ of orientation-preserving piecewise integral projective homeomorphisms of $\triangle_n$. Then $PIP^+(\triangle_n)$ is a group, and is a subgroup of $PIP(\triangle_n)$ of index 2.

We next investigate $PIP^+(\triangle_1)$. Let $\triangle'_1$ be the 1-simplex in $\mathbf{R}^2$ consisting of points $(t, 1)$ with t in the closed interval $[0, 1]$. The linear automorphism of $\mathbf{R}^2$ which maps $(1, 0)$ to $(1, 1)$ and $(0, 1)$ to $(0, 1)$ induces a homeomorphism from $\triangle_1$ to $\triangle'_1$. This linear automorphism is given by a matrix in $SL(2, \mathbf{Z})$. Thus we can "conjugate" the above discussion leading to the definition of $PIP^+(\triangle_1)$ to $\triangle'_1$: we get a group $PIP^+(\triangle'_1)$ which is isomorphic to $PIP^+(\triangle_1)$. In so doing, ρ is replaced by the map ρ' that sends (x, y) to $(\frac{x}{y}, 1)$ if $y \neq 0$ and to $(0, 1)$ if $y = 0$. An integral projective map for $\triangle'_1$ is the composition of ρ' and a function induced by a matrix in $GL(2, \mathbf{Z})$. An integral subsimplex of $\triangle'_1$ is a subsimplex of $\triangle'_1$ which is homeomorphic to $\triangle'_1$ by a $\triangle'_1$-integral projective map.

Now we identify $[0, 1]$ with $\triangle'_1$ via the map $t \mapsto (t, 1)$. Let a be a nonnegative integer and let b, c, d be positive integers such that $a \leq b$ and $c \leq d$. Then $\gcd(a, b) = 1 = \gcd(c, d)$, $\frac{a}{b} < \frac{c}{d}$, and $\left[\frac{a}{b}, \frac{c}{d}\right]$ is an integral subsimplex of $[0, 1]$ if and only if $ad - bc = -1$. Suppose a, b, c, d are as above such that $\left[\frac{a}{b}, \frac{c}{d}\right]$ is an integral subsimplex of $[0, 1]$. By definition the *left part* of $\left[\frac{a}{b}, \frac{c}{d}\right]$ is $\left[\frac{a}{b}, \frac{a+c}{b+d}\right]$ and the *right part* of $\left[\frac{a}{b}, \frac{c}{d}\right]$ is $\left[\frac{a+c}{b+d}, \frac{c}{d}\right]$. The left and right parts of $\left[\frac{a}{b}, \frac{c}{d}\right]$ are integral subsimplices of $[0, 1]$. The *tree of integral subsimplices* of $[0, 1]$ is the tree $\mathcal{T}'$ with vertices the integral subsimplices of $[0, 1]$ and with edges the pairs (I, J) where I and J are integral subsimplices of $[0, 1]$ and I is either the left part of J or the right part of J. An edge (I, J) of $\mathcal{T}'$ is a *left edge* if I is the left part of J and is a *right edge* if I is the right part of J. If we replace each vertex $\left[\frac{a}{b}, \frac{c}{d}\right]$

of $\mathcal{T}'$ by the Farey mediant $\frac{a+c}{b+d}$ of $\frac{a}{b}$ and $\frac{c}{d}$ and keep the same incidence relation, then $\mathcal{T}'$ becomes the Farey tree.

To see that $\mathcal{T}'$ is connected, let a be a nonnegative integer and let b, c, d be positive integers such that $\gcd(a,b) = 1 = \gcd(c,d)$, $\left[\frac{a}{b}, \frac{c}{d}\right] \neq [0,1]$, and $\left[\frac{a}{b}, \frac{c}{d}\right]$ is an integral subsimplex of $[0,1]$. First suppose that $a < c$. Let $r = c-a$ and let $s = d-b$. Then $-1 = ad-bc = a(b+s)-b(a+r) = as-br$, so $as = ar + (b-a)r - 1$, which implies that $s \geq r$. Furthermore, $\left[\frac{a}{b}, \frac{r}{s}\right]$ is an integral subsimplex of $[0,1]$ and $\left[\frac{a}{b}, \frac{c}{d}\right]$ is the left part of $\left[\frac{a}{b}, \frac{r}{s}\right]$. Now suppose that $a > c$. Let $r = a-c$ and let $s = b-d$. Then $-1 = ad - bc = (c+r)d - (d+s)c = dr - cs$, so $cs = rd + 1$ and $s > r$. Furthermore, $\left[\frac{r}{s}, \frac{c}{d}\right]$ is an integral subsimplex of $[0,1]$ and $\left[\frac{a}{b}, \frac{c}{d}\right]$ is the right part of $\left[\frac{r}{s}, \frac{c}{d}\right]$. If $a = c$, then $a = c = 1$, $b = d+1$, and $\left[\frac{a}{b}, \frac{c}{d}\right]$ is the right part of $\left[\frac{0}{1}, \frac{1}{d}\right]$. It follows that $\mathcal{T}'$ is connected and hence $\mathcal{T}'$ is an ordered rooted binary tree.

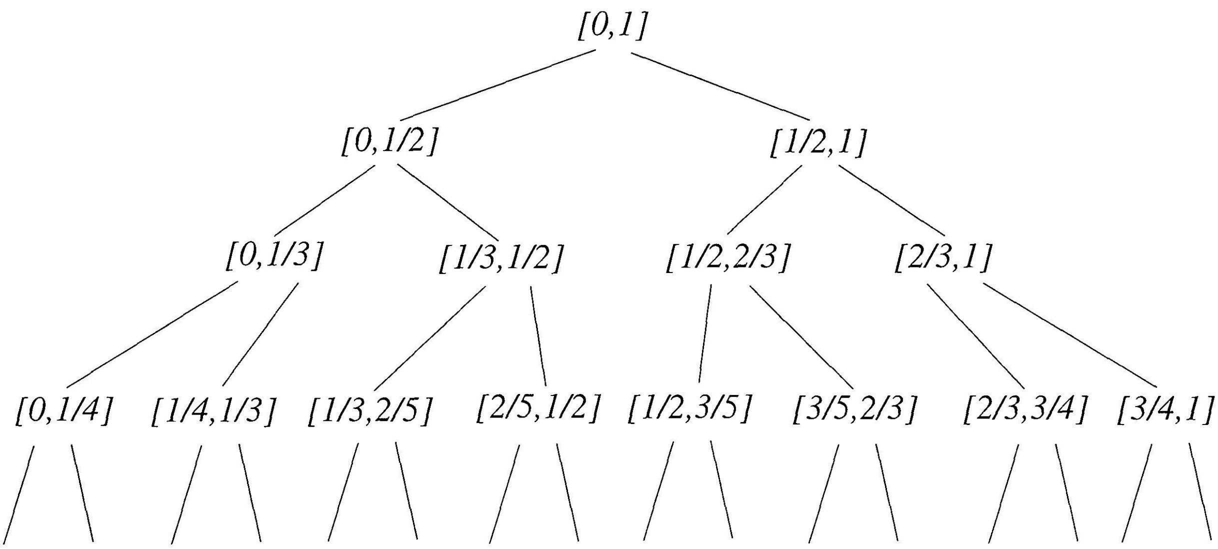

FIGURE 20

The tree $\mathcal{T}'$ of integral subsimplices of $[0,1]$

Now we consider integral projective maps for $[0,1]$. It is easy to see that they are given by linear fractional transformations corresponding to matrices in $GL(2,\mathbf{Z})$. Let $\left[\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right]$ and $\left[\frac{a}{b}, \frac{c}{d}\right]$ be integral subsimplices of $[0,1]$ as above. There is a unique integral projective map $f \colon \left[\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right] \to \left[\frac{a}{b}, \frac{c}{d}\right]$ with $f\left(\frac{\alpha}{\beta}\right) = \frac{a}{b}$ and $f\left(\frac{\gamma}{\delta}\right) = \frac{c}{d}$. The function f is defined by

$$f(t) = \frac{(c\beta - a\delta)t + (a\gamma - c\alpha)}{(d\beta - b\delta)t + (b\gamma - d\alpha)}$$

as a linear fractional transformation and is given by the matrix

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}^{-1}.$$

Since

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha+\gamma \\ \beta+\delta \end{pmatrix},$$

it follows that $f(\frac{\alpha+\gamma}{\beta+\delta}) = \frac{a+c}{b+d}$, and hence $f\left(\left[\frac{\alpha}{\beta}, \frac{\alpha+\gamma}{\beta+\delta}\right]\right) = \left[\frac{a}{b}, \frac{a+c}{b+d}\right]$ and $f\left(\left[\frac{\alpha+\gamma}{\beta+\delta}, \frac{\gamma}{\delta}\right]\right) = \left[\frac{a+c}{b+d}, \frac{c}{d}\right]$. This shows that an integral projective map $f : \left[\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right] \to \left[\frac{a}{b}, \frac{c}{d}\right]$ restricts to integral projective maps

$$f| : \left[\frac{\alpha}{\beta}, \frac{\alpha+\gamma}{\beta+\delta}\right] \to \left[\frac{a}{b}, \frac{a+c}{b+d}\right] \quad \text{and} \quad f| : \left[\frac{\alpha+\gamma}{\beta+\delta}, \frac{\gamma}{\delta}\right] \to \left[\frac{a+c}{b+d}, \frac{c}{d}\right].$$

The converse is also true; if

$$g_1 : \left[\frac{\alpha}{\beta}, \frac{\alpha+\gamma}{\beta+\delta}\right] \to \left[\frac{a}{b}, \frac{a+c}{b+d}\right] \quad \text{and} \quad g_2 : \left[\frac{\alpha+\gamma}{\beta+\delta}, \frac{\gamma}{\delta}\right] \to \left[\frac{a+c}{b+d}, \frac{c}{d}\right]$$

are integral projective maps, then they are the restrictions of an integral projective map $g : \left[\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right] \to \left[\frac{a}{b}, \frac{c}{d}\right]$. It follows as in §2 that there is a bijection between $PIP^+(\triangle_1)$ and the set of reduced tree diagrams.

Suppose $f, g \in PIP^+(\triangle_1)$, and let (P, Q) and (R, S) be reduced tree diagrams for f and g. Let Q' be a $\mathcal{T}'$-tree such that $Q \subset Q'$ and $R \subset Q'$. Then there are $\mathcal{T}'$-trees P' and S' such that $P \subset P'$, $S \subset S'$, (P', Q') is a tree diagram for f and (Q', S') is a tree diagram for g. Then (P', S') is a tree diagram for gf. This implies that the group structure for $PIP^+(\triangle_1)$ can be determined by the tree diagrams. Since the tree $\mathcal{T}$ of standard dyadic intervals is isomorphic, as an ordered rooted binary tree, to the tree $\mathcal{T}'$, this proves the following.

THEOREM 7.2. $F \cong PIP^+(\triangle_1)$.

We still view S^1 as $[0, 1]$ with the endpoints identified. A *piecewise integral projective (PIP)* homeomorphism of S^1 is a homeomorphism $f: S^1 \to S^1$ such that there is an integral subdivision $\mathcal{S}$ of $[0, 1]$ with $f|_\sigma$ integral projective for each simplex σ of $\mathcal{S}$. We denote by $PIP^+(S^1)$ the group of orientation-preserving PIP homeomorphisms of S^1. The proof of Theorem 7.2 also proves Theorem 7.3.

THEOREM 7.3. $T \cong PIP^{+}(S^1)$.

The three functions in $PIP^{+}(S^1)$ corresponding to A, B, and C are the following.

$$A(t) = \begin{cases} \frac{t}{t+1}, & 0 \le t \le \frac{1}{2} \\ \frac{-t+1}{-5t+4}, & \frac{1}{2} \le t \le \frac{2}{3} \\ \frac{2t-1}{t}, & \frac{2}{3} \le t \le 1 \end{cases} \qquad B(t) = \begin{cases} t, & 0 \le t \le \frac{1}{2} \\ \frac{3t-1}{4t-1}, & \frac{1}{2} \le t \le \frac{2}{3} \\ \frac{-6t+5}{-11t+9}, & \frac{2}{3} \le t \le \frac{3}{4} \\ \frac{2t-1}{t}, & \frac{3}{4} \le t \le 1 \end{cases}$$

$$C(t) = \begin{cases} \frac{-3t+2}{-5t+3}, & 0 \le t \le \frac{1}{2} \\ \frac{2t-1}{t}, & \frac{1}{2} \le t \le \frac{2}{3} \\ \frac{5t-3}{7t-4}, & \frac{2}{3} \le t \le 1 \end{cases}$$

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