# 2. Smaller alphabets

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The six-letter alphabet $\mathcal{A}$ can still be reduced by building blocks of three moves. From Corollary (i) we learn that they must be of the form $(\tilde{\alpha}, \tilde{\gamma}, \tilde{\beta})$ with $\tilde{\xi} \in \{\xi, \overline{\xi}\}$. Only five of these actually do occur:

THEOREM 2. *Triples of elements of* $c$ *form a square-free sequence* $h$ *over the five-letter alphabet* $\{\mathrm{A}, \mathrm{B}, \Gamma, \Delta, \mathrm{E}\}$ *with*

$$\mathrm{A} := (\alpha, \overline{\gamma}, \beta), \quad \mathrm{B} := (\alpha, \gamma, \overline{\beta}),$$

$$\Gamma := (\overline{\alpha}, \gamma, \beta), \quad \Delta := (\alpha, \gamma, \beta), \quad \mathrm{E} := (\overline{\alpha}, \gamma, \overline{\beta}). \qquad \square$$

*Proof.* From Corollary (o) we know: if $\gamma$ occurs in $c$ with a bar, its neighbors must be in odd positions and consequently unbarred. All the other triples turn up, the sequence starting

$$h = (\mathrm{A}, \mathrm{B}, \mathrm{A}, \Gamma, \mathrm{A}, \mathrm{B}, \Delta, \mathrm{E}, \mathrm{A}, \mathrm{B}, \mathrm{A}, \Gamma, \mathrm{A}, \mathrm{E}, \Delta, \Gamma, \mathrm{A}, \mathrm{B}, \mathrm{A}, \Gamma, \mathrm{A}, \mathrm{B}, \Delta, \mathrm{E}, \ldots).$$

Clearly, $h$ is square-free, since any square would lead to a square in $c$ as well, contradicting Theorem 1. $\square$

REMARK. $h$ (and consequently $c$) is *not* strongly square-free; can you spot an abelian square ? (The existence of a strongly square-free string over a five-letter alphabet has been established by P.A.B. Pleasants [17, Theorem 2].) $\square$

Let me finally mention another instance of the TH to emerge as a microcosmos: it is known that the number of *states*, i.e. distributions of the discs among the three pegs, of the TH which can be reached from the initial state with all discs on peg 0, say, in and in no less than $\mu \in \mathbf{N}_0$ moves, is a power of 2, namely $2^{\beta(\mu)}$, where $\beta(\mu)$ is the number of non-zero bits of $\mu$ (see [13, Proposition 5]). ($2^{\beta(\mu)}$ also happens to be the number of odd entries in the $\mu$th row of Pascal's arithmetical triangle, as was realized by J. W. L. Glaisher [10, second § 14]; cf. [14, formula (4)].) Denoting $\beta(\mu) \bmod 2$ by $m_\mu$, we obtain the *Thue-Morse sequence*

$$m := (0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, \ldots),$$

which by the subsequent substitution of $\alpha$ for $(0, 1, 1)$, $\beta$ for $(0, 1)$, and $\gamma$ for $(0)$ leads to the square-free sequence

$$t := (\alpha, \beta, \gamma, \alpha, \gamma, \beta, \alpha, \beta, \gamma, \beta, \alpha, \ldots)$$

over the three-letter alphabet $\{\alpha, \beta, \gamma\}$. This is, of course, the smallest possible alphabet with an infinite square–free string (clearly, a square-free word over a two-letter alphabet will come to an end after three entries) with which the whole theory started in the work of Axel Thue [19, Satz 3], [20, Sätze 6, 7, 20].

Obviously, $t$ (as in fact any word with more than 7 elements over a three-letter alphabet) is not strongly square-free. Maybe TH sequences hold a clue for a more direct approach to the question (cf. [6]), if there is an infinite strongly square-free string over a four-letter alphabet, which has been answered positively by V. Keränen [16] employing a computer-aided proof. (An abelian square of length $2 \cdot 6$ in $h$ starts after position $6$.)

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