1. Normal Forms for Quaternary Cubic Forms

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1. Normal Forms for Quaternary Cubic Forms

1.1. *Normal Forms for Quaternary Cubic Forms Defining Non-Singular Cubic Surfaces.* Here, the result is as follows:

THEOREM 1. *Every homogeneous polynomial of degree 3 in four variables defining a non-singular cubic surface can be brought into one of the following canonical forms ($r_i, r, s, t \in \mathbf{C}^*$):*

$$(*) \qquad r_1x_1^3 + r_2x_2^3 + r_3x_3^3 + r_4x_4^3 + r_5(-x_1 - x_2 - x_3 - x_4)^3,$$

$$\text{where } \sum_{i=1}^{5} \pm 1/\sqrt{r_i} \neq 0 \qquad \textit{(Sylvester's pentahedral form)}$$

$$(*_1) \qquad r(x_1^3 + x_2^3 + x_3^3 + x_4^3) \qquad \textit{(diagonal form)}$$

$$(*_2) \qquad rx_1^3 + x_2^3 + x_3^3 + x_4^3 - 3sx_2x_3x_4,$$

$$\text{where } (s^3 - 1)(s^3 + 8) \neq 0 \qquad \textit{(non-equianharmonic form)}$$

$$(*_3) \qquad x_2^3 + x_3^3 + x_4^3 - 3x_1^2(r_2x_2 + r_3x_3 + r_4x_4)$$

$$(*_4) \qquad x_2^3 + x_3^3 + x_4^3 - 3x_1^2(r_1x_1 + r_2x_2 + r_3x_3 + r_4x_4)$$

$$(*_5) \qquad 2rx_1^3 + x_2^3 + x_3^3 - 3x_1(sx_1x_2 + x_1x_3 + tx_4^2),$$

$$\text{where } st(r \pm s^{\frac{3}{2}} \pm 1) \neq 0.$$

For a proof of this theorem, we refer the reader to Segre's book [Se]. We will also call a form being equivalent to a form of type $(*)$ a *Sylvestrian pentahedral form.* Such a form determines a configuration of five planes which is called the *Sylvester pentahedron.* Forms being equivalent to diagonal or non-equianharmonic forms will be called *degenerate Sylvestrian pentahedral forms.*

REMARK 1. Given a cubic form f defining a non-singular cubic surface, one is led to ask to which of the above forms f is equivalent. This problem is related to the geometry of the Hessian surface $H_f = 0$ in the following way:

If the Hessian surface is reducible, there are two possibilities: Either it consists of four different planes or of a cone over a smooth plane cubic and a plane. In the first case, f is equivalent to a diagonal form, and in the second case, f is equivalent to a non-equianharmonic form.

If the Hessian surface is irreducible, we have to look at its singularities. If there are precisely ten A_1-singularities, f is equivalent to a Sylvestrian pentahedral form, and the Sylvester pentahedron is determined by the configuration of the singular points. If there are seven singular points, one A_1-singularity and six A_k-singularities with $k \geq 2$, then f is equivalent to a form $(*_3)$ or $(*_4)$ depending on whether the intersection of the Hessian surface with the tangent cone to the A_1-singularity consists of a double line and an irreducible conic or of a double line and a reducible conic. If there are four singular points on the Hessian surface, then f can be brought into a form of type $(*_5)$. In any case, much information on the canonical form can be read off the configuration of the singular points of $H_f = 0$. We refer the reader to [Se] and [Sch1] for the details.

The following results on canonical forms of quaternary cubic forms can be easily derived from the treatment of Bruce and Wall [BW] of the classification of singular cubic surfaces.

1.2. *Normal Forms for Quaternary Cubic Forms Defining Cubic Surfaces with Isolated Singularities.* Here, the normal form of f depends on the configuration of the singularities on the surface $f = 0$, and we obtain:

THEOREM 2. *The table overleaf lists the normal forms for quaternary cubic forms defining cubic surfaces with isolated singularities. The configuration of singularities on the respective surface is noted in the first column. Here, A_1 etc. refer to the classification of singularities (see e.g. [AGV], 242ff). Thus, $2A_1A_2$ means that there are two A_1-singularities and one A_2-singularity on the respective surface. It is assumed throughout that $l \in \mathbf{C}^*$.*

REMARK 2. The two different forms with a D_4-singularity are again distinguished by the geometry of the Hessian surface: The Hessian surface consists in the first case of a double plane and an irreducible quadric cone and in the second case of a double plane and two simple planes.

A_1	$lx_4(x_2^2 - x_1x_3) +$ $+ x_2\big(x_1 - (1+\rho_1)x_2 + \rho_1x_3\big)\big(x_1 - (\rho_2+\rho_3)x_2 + \rho_2\rho_3x_3\big),$ $\rho_i \in \mathbf{C} \setminus \{0,1\}$ *pairwise different*
$2A_1$	$lx_4(x_2^2 - x_1x_3) + x_2\big(x_1 - (1+\rho_1)x_2 + \rho_1x_3\big)(x_1 - \rho_2x_2),$ $\rho_i \in \mathbf{C} \setminus \{0,1\}$ *not equal*
$3A_1$	$lx_4(x_2^2 - x_1x_3) + x_2^2\big(x_1 - (1+\rho)x_2 + \rho x_3\big),\ \ \rho \in \mathbf{C} \setminus \{0,1\}$
$4A_1$	$lx_4(x_2^2 - x_1x_3) + x_2^2(x_1 - 2x_2 + x_3)$
A_1A_2	$lx_4(x_2^2 - x_1x_3) + x_1x_2\big(x_1 - (1+\rho)x_2 + \rho x_3\big),$ $\rho \in \mathbf{C} \setminus \{0,1\}$
$2A_1A_2$	$lx_4(x_2^2 - x_1x_3) + x_2^2(x_1 - x_2)$
A_12A_2	$lx_4(x_2^2 - x_1x_3) + x_2^3$
A_1A_3	$lx_4(x_2^2 - x_1x_3) + x_1^2x_2 - x_1x_2^2$
$2A_1A_3$	$lx_4(x_2^2 - x_1x_3) + x_1x_2^2$
A_1A_4	$lx_4(x_2^2 - x_1x_3) + x_1^2x_2$
A_1A_5	$lx_4(x_2^2 - x_1x_3) + x_1^3$
A_2	$lx_4x_1x_2 - x_3(x_1^2 + x_2^2 + x_3^2 + \rho_1x_1x_3 + \rho_2x_2x_3),$ $\rho_1, \rho_2 \in \mathbf{C} \setminus \{-2, +2\}$
$2A_2$	$lx_4x_1x_2 - x_3(x_1^2 + x_3^2 + \rho x_1x_3),\ \ \rho \in \mathbf{C} \setminus \{-2, +2\}$
$3A_2$	$lx_4x_1x_2 - x_3^3$
A_3	$lx_4x_1x_2 + x_1(x_1^2 - x_3^2) + \rho x_2(x_2^2 - x_3^2),\ \ \rho \in \mathbf{C}^*$
A_4	$lx_4x_1x_2 + x_1^2x_3 + x_2(x_2^2 - x_3^2)$
A_5	$lx_4x_1x_2 + x_1^3 + x_2(x_2^2 - x_3^2)$
D_4^I	$lx_4x_1^2 + x_2^3 + x_3^3 + x_1x_2x_3$
D_4^{II}	$x_4x_1^2 + x_2^3 + x_3^3$
D_5	$x_4x_1^2 + x_1x_3^2 + x_2^2x_3$
E_6	$x_4x_1^2 + x_1x_3^2 + x_2^3$
$\widetilde{E}_6$	$x_1^3 + x_2^3 + x_3^3 - 3lx_1x_2x_3, \quad l^3 \neq 1$

1.3. *Normal Forms for Quaternary Cubic Forms Defining Irreducible Cubic Surfaces with Non-Isolated Singularities.*

PROPOSITION 1. *The canonical forms for quaternary cubic forms defining irreducible cubic surfaces with non-isolated singularities are the following:*

Canonical form	*The surface* $f = 0$
$x_1^2x_3 + x_2^2x_4$	*Whitney's ruled surface*
$x_1^2x_3 + x_1x_2x_4 + x_2^3$	*Cayley's ruled surface*
$x_1x_3^2 + x_1x_2^2 + x_2^3$	*Cone over a nodal cubic*
$x_1^2x_3 + x_2^3$	*Cone over Neil's parabola*

REMARK 3. Cayley's ruled surface is actually a degeneration of Whitney's surface. Explicit constructions of those surfaces can be found in [Ha1], 330f, for Whitney's surface and in [Ha2], 80, for Cayley's surface.

1.4. *Normal Forms for Quaternary Cubic Forms Defining Reducible Cubic Surfaces.* Here, one obtains the following obvious result:

PROPOSITION 2. *A quaternary cubic form defining a reducible cubic surface can be brought into one of the following canonical forms:*

Canonical form	*The surface* $f = 0$
$(x_1 + x_2)(x_1x_2 + x_3x_4)$	*Non-sing. quadric w. transversal plane*
$x_1(x_1x_2 + x_3x_4)$	*Non-sing. quadric w. tangent plane*
$x_1(x_2^2 + x_3x_4)$	*Quadric cone w. transversal plane*
$x_2(x_2^2 + x_3x_4)$	*Cone over plane conic w. transversal line*
$x_3(x_2^2 + x_3x_4)$	*Cone over plane conic w. tangent*
$x_1x_2x_3$	*Three different planes*
$x_1x_2(x_1 + x_2)$	*Three different planes in a pencil*
$x_1^2x_2$	*Double plane and simple plane*
x_1^3	*Triple plane*