2. The Invariant Theory of Quaternary Cubic Forms

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2. The Invariant Theory of Quaternary Cubic Forms

2.1. *Stable, Semistable and Nullforms.* The stable and semistable quaternary cubic forms and the quaternary cubic nullforms were determined by Hilbert [Hi] (for the definition of semistable and stable see [Ne], *nullform* means non-semistable form):

THEOREM 3. i) *A quaternary cubic form f is stable (resp. semistable) if and only if the surface $\{f = 0\}$ has at most singularities of type A_1 (resp. A_2).*

ii) *A quaternary cubic form f is a nullform if and only if the surface $\{f = 0\}$ has isolated singularities of type A_k $(k \geq 3)$, D_4, D_5, E_6, or $\widetilde{E}_6$, or if it has non-isolated singularities.*

2.2. *Degenerations of Orbits of Semistable Forms.* First, one observes that the semistable forms with closed orbit are precisely the forms whose associated cubic surfaces have three A_2-singularities. Applying Luna's slice theorem, one then computes the following table of degenerations where we characterize a form by the configuration of singularities on the corresponding cubic surface:

$$\begin{array}{ccccc}
A_2 & & A_2A_1 & & 2A_1A_2 \\
 & \searrow & & \searrow & \downarrow \\
 & & 2A_2 & & 2A_2A_1 \\
 & & & \searrow & \downarrow \\
 & & & & 3A_2
\end{array}$$

The details can be found in [Sch1], 58ff.

2.3. *The Ring of Invariants.* Proofs of the following results can be found in the paper [Be]. We want to describe the ring $A := \mathbf{C}[S^3(\mathbf{C}^{4^\vee})]^{\mathrm{SL}_4(\mathbf{C})}$. This is the coordinate ring of the categorical quotient $S^3(\mathbf{C}^{4^\vee})/\!/\,\mathrm{SL}_4(\mathbf{C})$. It is the ring of polynomial expressions in the coefficients of cubic polynomials which are constant on all $\mathrm{SL}_4(\mathbf{C})$-orbits. In order to describe the ring A, we first introduce the following vector space

$$S := \left\{ r_1x_1^3 + r_2x_2^3 + r_3x_3^3 + r_4x_4^3 + r_5x_5^3 \mid \sum x_i = 0 \right\}.$$

On S, there is a natural action of the alternating group $\mathfrak{A}_5$, and $A \subset \mathbf{C}[S]^{\mathfrak{A}_5}$. This inclusion is constructed as follows: The group of automorphisms H of the Sylvester pentrahedron naturally acts on S, and it can be shown that the natural morphism $S/\!/H \longrightarrow S^3(\mathbf{C}^{4\vee})/\!/\mathrm{SL}_4(\mathbf{C})$ is birational. This induces the inclusion $A \subset \mathbf{C}[S]^H$. Now, H is a finite group of order 480 obviously containing $\mathfrak{A}_5$. Denote by σ_i, $i = 1,2,3,4,5$, and v the i-th symmetric function and the Vandermonde determinant in the r_i. Then $\mathbf{C}[S]^{\mathfrak{A}_5} = \mathbf{C}[\sigma_1, \dots, \sigma_5, v]$.

THEOREM 4. *The ring of invariants A is the subring of $\mathbf{C}[S]^{\mathfrak{A}_5}$ generated by the following invariant polynomials*

$$I_8 := \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} := \sigma_5^3\sigma_1, \quad I_{24} := \sigma_5^4\sigma_4,$$
$$I_{32} := \sigma_5^6\sigma_2, \quad I_{40} := \sigma_5^8, \quad I_{100} := \sigma_5^{18}v,$$

which satisfy a relation

$$I_{100}^2 = P(I_8, I_{16}, I_{24}, I_{32}, I_{40}).$$

2.4. *The Discriminant.* Using techniques from the paper [BC], one obtains the following

PROPOSITION 3. *The discriminant of quaternary cubic forms is given by the formula*

$$\Delta = (I_8^2 - 64 I_{16})^2 - 2^{11}(I_8 I_{24} + 8 I_{32}).$$

2.5. *Moduli Spaces of Cubic Surfaces.* Define $\overline{\mathcal{M}}$ to be the hypersurface $\{I_{100}^2 - P(I_8, I_{16}, I_{24}, I_{32}, I_{40}) = 0\}$ in the weighted projective space $\mathbf{P}(8,16,24,32,40) = \mathbf{P}(1,2,3,4,5)$. Then $\mathcal{M} := \overline{\mathcal{M}} \setminus \{\Delta = 0\}$ is a moduli space for non-singular cubic surfaces. On the other hand, every non-singular cubic surface can be obtained as the blow up of $\mathbf{P}_2$ in six points in general position. The sextuples of points in general position form an open subset $\mathcal{U} \subset S^6\mathbf{P}_2$ of the sixth symmetric power of $\mathbf{P}_2$. Furthermore, there is an action of $\mathrm{PGL}_3(\mathbf{C})$ on $\mathcal{U}$, and the geometric quotient $\mathcal{N} := \mathcal{U}/\!/\mathrm{PGL}_3(\mathbf{C})$ does exist [Is]. By [Is], §6, $\mathcal{N}$ is a coarse moduli space for pairs (X, L) consisting of a cubic surface X and a globally generated line bundle L which defines a blow down $X \longrightarrow \mathbf{P}_2$. Forgetting the line bundle L provides us with a morphism $\mathcal{N} \longrightarrow \mathcal{M}$, so that there is a surjection $f\colon \mathcal{U} \longrightarrow \mathcal{M}$. Hence, we can view the invariants of quaternary cubic forms as regular functions on $\mathcal{U}$. This relates the geometry of the cubic surface to the set of six points. One obtains, e.g., PROPOSITION 4. *The set of sextuples in $\mathcal{U}$ whose associated cubic surface is given by an equation which is not a (nondegenerate) Sylvestrian pentahedral form is the Zariski-closed subset* $\{f^*I_{40} = 0\}$.

Of course, a better understanding of the geometric meaning of the other invariants should allow to extend this result.

II. CUBIC FORMS OF PROJECTIVE THREEFOLDS

1. PRELIMINARIES

For the convenience of the reader, we have collected the crucial theorems which we will use in the construction of our examples.

1.1. *The Lefschetz Theorem on Hyperplane Sections.* We summarize Bertini's Theorem and Lefschetz' Theorem in:

THEOREM 5. *Let Y be a projective manifold, L a very ample line bundle on Y, and $X := Z(s)$ the zero-set of a general section $s \in H^0(X, L)$. Then X is a manifold (connected if* $\dim Y \geq 2$*), and the inclusion $\iota \colon X \hookrightarrow Y$ induces isomorphisms*

$$\iota^* : H^i(Y, \mathbf{Z}) \longrightarrow H^i(X, \mathbf{Z}), \qquad i = 1, \ldots, \dim Y - 2;$$

$$\iota_* : \pi_i(X) \longrightarrow \pi_i(Y), \qquad i = 1, \ldots, \dim Y - 2.$$

Proof. [La], Th. 3.6.7 & Th. 8.1.1. □

1.2. *Formulas for Blow Ups.* A very simple way to obtain a new manifold from a given one is the blow up in a point or along a smooth curve. The cup form behaves as follows (we will suppose for simplicity that $H^2(Y, \mathbf{Z})$ is without torsion):

THEOREM 6. i) *Let $\sigma \colon X \longrightarrow Y$ be the blow up of Y in a point. Let $q(x_1, \ldots, x_n)$ be the cubic polynomial which describes the cup form of Y w. r. t. the basis $(\kappa_1, \ldots, \kappa_n)$ of $H^2(Y, \mathbf{Z})$. If $h_0 \in H^2(X, \mathbf{Z})$ is the cohomology class of the exceptional divisor, then $(h_0, \sigma^*\kappa_1, \ldots, \sigma^*\kappa_n)$ is a basis of $H^2(Y, \mathbf{Z})$ w. r. t. which the cup form of X is given by*

$$x_0^3 + q(x_1, \ldots, x_n).$$