4. Deus ex machina: l_2-cohomology

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **43 (1997)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.05.2024**

PROPOSITION 1. *Let M be a complex surface, and assume that its fundamental group G fulfills $p(G) \geq 0$. Then the holomorphic Euler characteristic of M is ≥ 0.*

By the Kodaira-Enriques classification it follows that M cannot be ruled over a curve of genus ≥ 2.

REMARK. The formulae above leading to the holomorphic Euler characteristic refer to the orientation of the complex surface dictated by the complex structure. Thus the argument is valid only if in *that* orientation $\sigma(M) \leq 0$. If however $\sigma(M) > 0$ then $p(G) \geq 0$ implies that $2 - 2\beta_1(G) + 2\beta_2^+{}_{\text{wrong}}(M) \geq 0$ where $\beta_2^+{}_{\text{wrong}}$ refers to the "wrong" orientation and is $= \beta_2^-(M)$. Now $\beta_2^+(M) > \beta_2^-(M)$ by assumption. Thus the result remains true; the holomorphic characteristic is > 0.

III) *Donaldson Theory.* Finitely presented groups G with $p(G) \geq 0$ and $\beta_1(G) \geq 4$ do not qualify for the Theorems A,B, and C of Donaldson [D] relating to non-simply connected topological manifolds. Indeed in these theorems the signature is assumed to be negative with $\beta_2^+ = 0$, 1 or 2. However $p(G) \geq 0$ means $2 - 2\beta_1(G) + 2\beta_2^+(M) \geq 0$, i.e. $\beta_2^+(M) \geq \beta_1(G) - 1$.

4. DEUS EX MACHINA: l_2-COHOMOLOGY

4.1. We recall in a few words the (cellular) definition of l_2-cohomology and l_2-Betti numbers, in the case of a 4-manifold M but things apply to any finite cell-complex.

Some definitions: For any countable group G let l_2G be the Hilbert space of square-integrable real functions on G, with G operating on the left, and NG the algebra of bounded G-equivariant linear operators on l_2G. A Hilbert-G-module H is a Hilbert space with isometric left G-action which admits an isometric G-equivariant imbedding into some l_2G^m (direct sum of m copies of l_2G). The projection operator ϕ of l_2G^m with image H is given by a matrix (ϕ_{kl}), $\phi_{kl} \in NG$. The "trace" $\sum \langle \phi_{kk}(1), 1 \rangle$ is the von Neumann dimension $\dim_G H$; it is a real number ≥ 0, and $= 0$ if and only if $H = 0$.

Let $\widetilde{M}$ be the universal cover of M with the cell-decomposition corresponding to that chosen in M. The square-integrable real i-cochains of $\widetilde{M}$ constitute a Hilbert space $C^i_{(2)}(\widetilde{M})$ with isometric G-action. It decomposes into the direct sum of α_i copies of l_2G, $i = 0, \ldots, 4$. As before α_i denotes the number of i-cells of M; G is the fundamental group of M acting by permutation of the cells of $\widetilde{M}$. The $C^i_{(2)}$ with the induced coboundary operators form a Hilbert-G-module chain complex. The cohomology H^i of that complex is easily identified with $H^i(M; l_2G)$, cohomology with local coefficients (see, e.g. [E2]). The *reduced* cohomology group $\overline{H}^i$ (i.e. cocycles modulo the closure of coboundaries) of that complex can be imbedded in $C^i_{(2)}$ as a G-invariant subspace and is therefore a Hilbert-G-module. Its von Neumann dimension $\dim_G \overline{H}^i(\widetilde{M})$ is the i-th l_2-Betti number $\overline{\beta}_i(M)$. It is a topological, even a homotopy, invariant of M.

4.2. Since $\dim_G C^i_{(2)} = \alpha_i$ and since the von Neumann dimension behaves like a rank, the usual Euler-Poincaré argument shows that the l_2-Betti numbers compute the Euler characteristic exactly as the ordinary Betti numbers do:

$$\chi(M) = \sum (-1)^i \overline{\beta}_i(M) .$$

Moreover the $\overline{\beta}_i$ of a closed manifold fulfill Poincaré duality. Thus

$$\chi(M) = 2\overline{\beta}_0 - 2\overline{\beta}_1 + \overline{\beta}_2 .$$

According to Atiyah's l_2-signature theorem [A], $\sigma(M)$ can also be expressed by appropriate l_2-Betti numbers: $\overline{H}^2(\widetilde{M})$ splits into two complementary G-invariant subspaces with von Neumann dimensions $\overline{\beta}_2^+(M)$ and $\overline{\beta}_2^-(M)$, and $\sigma(M)$ is their difference. Thus, as with ordinary Betti numbers, one has

$$\chi(M) + \sigma(M) = 2\overline{\beta}_0(G) - 2\overline{\beta}_1(G) + 2\overline{\beta}_2^+(M) .$$

We now assume G to be infinite. Then $\overline{\beta}_0(G) = 0$. Indeed a 0-cocycle f in $\widetilde{M}$ is a constant and if $\widetilde{M}$ is an infinite complex f can be l_2 only if it is $= 0$.

THEOREM 2. *If for a finitely presented group G the first l_2-Betti number $\overline{\beta}_1(G)$ is 0 then the invariants $p(G)$ and $q(G)$ are non-negative.*

COROLLARY 3. *If $\overline{\beta}_1(G) = 0$ then* $\operatorname{def}(G) \leq 1$.

COROLLARY 4. *If $G = \pi_1$(complex surface M) with $\overline{\beta}_1(G) = 0$ then the holomorphic Euler characteristic of M is non-negative.*

4.3. There are many groups for which it is known that $\overline{\beta}_1(G) = 0$. A good list is given in [B-V]. We mention here three big and interesting classes of groups with that property.

1) All finitely generated amenable groups [C-G]. We recall that this class includes the virtually solvable groups, thus in particular the finitely generated Abelian groups (whence $\mathbf{Z}^n$, example 1) in 2.2). [Actually for an amenable group G with $K(G,1)$ of finite type, i.e. there is a $K(G,1)$ with finite m-skeleta, all l_2-Betti numbers are 0.]

THEOREM 5. *If G is a finitely presented amenable group then $p(G)$ and $q(G)$ are non-negative.*

2) [L1] All finitely presented groups G containing an infinite finitely generated normal subgroup N such that there is in G/N an element of infinite order. For these "Lück groups" one has the same conclusions as in the amenable case. — In [L1] the subgroup N is assumed to be finitely presented. Lück has shown later [L2] that the weaker assumption above is sufficient.

3) The statement of Theorem 5 also holds more generally for a finitely presented group G which contains a finitely generated normal subgroup N such that G/N is infinite and amenable [E2]. The proof is somewhat different: It makes use not of the universal cover but of the cover belonging to N. The amenable group G/N operates on that cover and one can use the l_2-Betti numbers relative to G/N. — A simple example is given by a group with finitely generated commutator subgroup and infinite Abelianisation.

4.4. REMARKS.

1) We note that for finitely presented infinite amenable groups, and also for groups as in 4.3, 3) above, the deficiency is ≤ 1. This can also be proved without 4-manifolds: It suffices to consider a $K(G,1)$ with 2-skeleton corresponding to a presentation of G.

2) It is well-known that a group with deficiency ≥ 2 cannot be amenable since it contains free subgroups of rank ≥ 2; see [B-P], where a stronger result is proved.

3) There is a class of groups for which $\overline{\beta}_1$ is positive: The groups G with infinitely many ends (i.e. with $H^1(G;\mathbf{Z}G)$ of infinite rank; here one takes ordinary cohomology with local coefficients). A nice proof for this can be found in [B-V]. Another approach is to use Stallings' structure theorem from which it follows that these groups contain free subgroups of rank ≥ 2 and thus are non-amenable. For non-amenable groups the Guichardet amenability criterion [G] tells that $\overline{H}^1(G;l_2G) = H^1(G;l_2G)$. The coefficient

map $H^1(G; \mathbf{Z}G) \longrightarrow H^1(G; l_2G)$ induced by the imbedding $\mathbf{Z}G \longrightarrow l_2G$ is easily seen to be injective. Since we have assumed $H^1(G; \mathbf{Z}G) \neq 0$ the result follows.

5. The vanishing of $q(G)$

5.1. Here we mention in a few words what happens when for a finitely presented group G the invariant $q(G)$ is 0. For the details and more comments we refer to the paper [E2]. We thus consider a 4-manifold M with $\pi_1(M) = G$ and $\chi(M) = 0$.

Since we restrict attention to groups with $\overline{\beta}_1(G) = 0$ the vanishing of $\chi(M)$ implies $\overline{\beta}_2(M) = 0$, whence $\overline{H}^2(\widetilde{M}) = 0$. As shown in [E2] by a spectral sequence argument it follows that $H^2(M; \mathbf{Z}G)$ is isomorphic to $H^2(G; \mathbf{Z}G)$, ordinary cohomology with local coefficients $\mathbf{Z}G$. By Poincaré duality $H^2(M; \mathbf{Z}G) = H_2(M; \mathbf{Z}G)$ which can be identified with $H_2(\widetilde{M}; \mathbf{Z})$. Since $\widetilde{M}$ is simply connected, $H_2(\widetilde{M}; \mathbf{Z})$ is isomorphic to the second homotopy group $\pi_2(\widetilde{M}) = \pi_2(M)$.

What about $H_3(\widetilde{M}; \mathbf{Z})$? It can be identified with $H_3(M; \mathbf{Z}G)$ which, by Poincaré duality, is $\cong H^1(M; \mathbf{Z}G) = H^1(G; \mathbf{Z}G)$. This group, the "endpoint-group" of G, is known to be either 0 or $\mathbf{Z}$ or of infinite rank. As mentioned in 4.4, remark 3) the latter case is excluded by our assumption $\overline{\beta}_1(G) = 0$. The case $H^1(G; \mathbf{Z}G) = \mathbf{Z}$ is exceptional: it means that G is virtually infinite cyclic, and we exclude this. Then $H_3(\widetilde{M}; \mathbf{Z}) = 0$.

5.2. We now add the assumption that $H^2(G; \mathbf{Z}G) = 0$. This is a property shared by many groups (e.g. duality groups). Then the homology groups $H_i(\widetilde{M}; \mathbf{Z})$ are $= 0$ for $i = 1, 2, 3, 4$ ($i = 4$ because $\widetilde{M}$ is an open manifold). Thus all homotopy groups of $\widetilde{M}$ are $= 0$, $\widetilde{M}$ is contractible, M is a $K(G, 1)$, and the group G fulfills Poincaré duality.

THEOREM 6. *Let G be an infinite, finitely presented group, not virtually infinite cyclic, fulfilling $\overline{\beta}_1(G) = 0$ and $H^2(G; \mathbf{Z}G) = 0$, and let M be a manifold with fundamental group G. If the Euler characteristic $\chi(M) = 0$, then M is an Eilenberg-MacLane space for G and G is a Poincaré duality group of dimension* 4.

We recall that for knot groups and 2-knot groups $q(G) = 0$, see examples 3) and 4) in 2.2. Theorem 6 can only be applied to 2-knot groups which are not classical knot groups since the latter have cohomological dimension 2.