NONMINIMAL RATIONAL CURVES ON K3 SURFACES

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INTRODUCTION

The following assertion was made, in 1943, by B. Segre ([Se]):

(**S**) *The general quartic surface F contains a finite number $c_h > 0$ of unicursal (i.e., rational) curves of degree 4h (for $h = 1, 2, 3, \ldots$).*

This was prompted by the opposite claim made by W. Fr. Meyer at the turn of the century ([Me], §3, pp. 1545–47):

(**M**) *On a generic (quartic surface) F_4 there can lie no (rational curve) R_m ($m = 1, 2, \ldots$).*

The notation R_m was commonly used to mean: *a rational curve of degree m*, but it is not clear whether Meyer intended to limit his statement to *smooth* rational curves. In fact, the argument he gives in support of his claim makes reasonable sense for smooth curves: it takes $4m + 1$ conditions to express that a quartic surface contains a smooth rational curve of degree m, but these curves depend on $4m$ constants only. Indeed, a parametrization $\varphi \colon \mathbf{P}^1 \to \mathbf{P}^3$ defined by four homogeneous polynomials of degree m depends on $4(m+1)$ coefficients, which are arbitrary up to multiplication by a common scalar; and the ∞^3 automorphisms of $\mathbf{P}^1$ preserve the image of such a map.

The independence of the conditions so expressed would need to be thoroughly examined. But, with this interpretation, assertion (**M**) does hold and can be derived from a celebrated theorem of Max Noether, which is very well established (see [De], thm. 1.2): a generic quartic surface (in a specific

sense) has no other divisors than its hypersurface sections; and the arithmetic genus of such divisors is never zero.

Nevertheless, as Segre noticed, Meyer's statement is certainly false when singularities are allowed. Indeed, it is well-known that a general quartic surface in $\mathbf{P}^3$ has 3200 tritangent planes. Each of them meets the surface in a quartic with 3 double points, which has geometric genus zero.

It is interesting to compare with a similar statement proved in 1979 by Mumford and by Bogomolov (see [M-M]):

THEOREM (Bogomolov & Mumford). *Every algebraic K3 surface over* $\mathbf{C}$ *contains a singular rational curve and a pencil of singular elliptic curves.*

Actually, they proved that a complete linear system of curves of *minimal* arithmetic genus (greater than one) on the surface contains at least one irreducible rational member. For a smooth quartic in $\mathbf{P}^3$, which is a special case of K3 surface, this deals with the relatively uninteresting case where $h = 1$.

Assertion (**S**) would be easy to establish if we knew that *every* complete linear system of curves of arithmetic genus greater than one on a K3 surface has an irreducible rational member. The main innovation of this paper is that, for a restricted family of K3 surfaces, we show the existence of singular rational irreducible members in some complete linear systems which are not minimal.

More precisely, we give a proof of Segre's assertion (**S**) for $h = 2$ and $h = 3$ (Theorem 3.4). In §4 we establish a similar result for K3 surfaces in $\mathbf{P}^4$ (Theorem 4.1). However, for reasons explained before Lemma 2.5, we have not been able to prove assertion (**S**) for $h > 3$.

SCHOLIA. What happens in reality is somewhat surprising. Sometimes the problem seems to be very easy, and sometimes very hard. We shall try to explain here why this is so.

First we recall that the set of space curves of a given degree can be viewed as a variety, by a construction usually attributed to Chow, though much of the idea goes back to Cayley ([Ca]). (See [Sh] for a brief, but enlightening discussion, and [H-P] for an almost exhaustive treatment.)

In a few words, let $V \subset \mathbf{P}^N$ be a projective variety of dimension n. We denote by $\widetilde{\mathbf{P}^N}$ the dual space of $\mathbf{P}^N$ and consider the set $\Delta \subset (\widetilde{\mathbf{P}^N})^{n+1} \times V$ of all points $(h_0, \ldots, h_n, x)$ such that every hyperplane h_i contains x. One can prove (cf. [Sh], Chap. 1, §6) that Δ is a closed set whose projection in $(\widetilde{\mathbf{P}^N})^{n+1}$ is defined by a single equation F_V. The form F_V is called the *Cayley form* associated with V, and its coefficients are the coordinates of the

Chow point of V. If – instead of looking merely at varieties – one considers all cycles of a given degree m and dimension n, one shows that the Chow points form a projective algebraic variety. This is called the *Chow variety* of all cycles in $\mathbf{P}^N$ of degree m and dimension n.

SCHOLIUM 1. It is known that the Chow variety of space curves of degree m has an irreducible component $\mathcal{R}_m$, of dimension $4m$, whose general element is the Chow point of a smooth rational curve (*cf.* [Co2], Lemma 2.4). Moreover, any irreducible space curve with degree m and geometric genus zero belongs to it.

We denote by $\mathcal{F}_K$ the projective space (of dimension 34) parametrizing all quartic surfaces in $\mathbf{P}^3$. For each value of m, we can consider the incidence correspondence $\mathcal{I}_m \subset \mathcal{R}_m \times \mathcal{F}_K$ consisting of all pairs (γ, F) with $\langle \gamma \rangle \subset F$, where $\langle \gamma \rangle$ denotes the support of the 1-cycle whose Chow point is $\gamma \in \mathcal{R}_m$.

Now a smooth quartic curve Γ of genus 0 in $\mathbf{P}^3$ is contained in a unique quadric surface. From this it is easy to compute that it is contained in precisely ∞^{17} surfaces $F \in \mathcal{F}_K$. (The 17 conditions coming from the Bézout theorem are independent; indeed, given any set of 16 points on Γ, there is a union of two quadrics through them which does not contain Γ.) Thus the incidence correspondence $\mathcal{I}_4$ has dimension $16 + 17 = 33$ above some nonempty open subset of $\mathcal{R}_4$.

But if we look at the family of plane quartics in $\mathbf{P}^3$, we see that its dimension is $14 + 3 = 17$ (one more than the dimension of $\mathcal{R}_4$!). It can be shown that those having 3 double points form a family of dimension $(14 - 3) + 3 = 14$. Now, for a quartic surface to contain such a singular curve Γ, it is enough to impose 11 simple points and the 3 double points, since this also represents $11 + 2 \cdot 3 = 4 \cdot 4 + 1$ intersections. Thus Γ is contained in $\infty^{\geq 20}$ surfaces $F \in \mathcal{F}_K$. It follows that the incidence correspondence $\mathcal{I}_4$ has a component of dimension $\geq 14 + 20 = 34$ above the singular plane quartics.[1])

Hence not only is $\mathcal{I}_4$ reducible, but it has a component of larger dimension than its dimension over the generic point of the Chow variety $\mathcal{R}_4$!

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[1]) In fact, equality holds. The referee suggests the following argument: as a complete intersection, any plane quartic is arithmetically Cohen-Macaulay of arithmetic genus 3. Therefore it imposes exactly $h^0(\mathcal{O}_\Gamma(4)) = 14$ conditions on quartic surfaces.

There is also a family of rational curves of degree 4 and arithmetic genus 1, namely the rational reduced and irreducible quartic curves which are complete intersections of two quadric surfaces (and have a double point). These curves are contained in ∞^{18} surfaces of degree 4.

As a matter of fact, this family of curves $\mathcal{S} \subset \mathcal{R}_4$ has dimension 15. So, the fibres above $\mathcal{S}$ span a 33-dimensional variety, which is also a component of $\mathcal{I}_4$.

This explains why we can say that both Segre and Meyer were right, in some sense: they referred to the images in $\mathcal{F}_K$ of *different components of* $\mathcal{I}_m$.

We proceed with an informal discussion of the case $h = 2$, for which one can also get a pretty clear picture:

SCHOLIUM 2. We refer to Max Noether ([No], §17) for a discussion of space curves of degree 8. Noether uses several criteria[2]) to establish that a *general* smooth rational octic Γ is contained in no more than two independent quartic surfaces[3]) F_4. Thus the incidence correspondence $\mathcal{I}_8$ has dimension $32 + 1 = 33$ above some nonempty open subset of $\mathcal{R}_8$ and could not possibly map surjectively onto $\mathcal{F}_K$ if it were irreducible. This is in agreement with Meyer's assertion for degree 8.

However, the complete intersections of a quartic and a quadric have the right dimension (33, which is one more than $\dim \mathcal{R}_8$). Our task will consist in showing that those with 9 double points form a subfamily of $\mathcal{R}_8$ of dimension $33 - 9 = 24$. Again, since these curves are contained in ∞^{10} quartic surfaces, we have to do with a component of larger dimension than the one above the general point of $\mathcal{R}_8$. We will then show that this component maps onto a dense constructible subset of $\mathcal{F}_K$.

Here is yet another heuristic way to understand why the dimension is one more than normally expected: *In $\mathcal{I}_8$ one can obtain a curve Γ with 9 double points by imposing only 8 nodes.*

Indeed, any quadric passing through the 8 nodes and one more point of Γ has $2 \cdot 8 + 1 = 17$ intersections with Γ. Hence Γ is contained in an irreducible quadric Q_0.

Since we are moving in $\mathcal{I}_8 \subset \mathcal{R}_8 \times \mathcal{F}_K$, the divisor Γ is of type $(4, 4)$ on Q_0, hence of arithmetic genus 9. But Γ is rational and the assigned singularities are ordinary double points. Hence Γ automatically acquires a ninth singular point.

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[2]) For instance, on a smooth quartic F containing Γ, any other quartic surface F' through Γ cuts out a residual curve Γ'. The linear system $|\Gamma'|$ has dimension 0. Indeed, $p_a(\Gamma) = 0 \Rightarrow (\Gamma)^2 = -2$; whence $(\Gamma')^2 = (\mathcal{O}(4) - \Gamma)^2 = -2$, so that Γ' is an isolated divisor. Hence any other quartic through Γ belongs to the pencil generated by F and F'.

[3]) One cannot leave out the word 'general'. Indeed one also finds some smooth rational curves of degree 8 on any smooth quadric, where they correspond to the divisors of type $(1, 7)$. These curves are therefore contained in ∞^9 (reducible) surfaces of degree 4.

As a matter of fact, this family of curves $\mathcal{S} \subset \mathcal{R}_8$ has dimension 24. So the fibres above $\mathcal{S}$ span a 33-dimensional variety, which is also a component of $\mathcal{I}_8$.

1. ABOUT FINITENESS

Segre's heuristic starting point was that a complete linear system of arithmetic genus g on a smooth quartic is also of dimension g. Now, demanding that the system have a member with g ordinary double points should represent g independent conditions. Hence, for every positive integer h, there should exist a finite number of rational irreducible curves in the complete linear system cut out by the family of all surfaces of degree h.

This argument is unsatisfactory as it stands, because there are always infinitely many *reducible* curves with the right number of nodes (unless $h = 1$). For instance, for $h = 2$ the curve cut by the union of a tritangent plane and any one of the infinitely many *bitangent* planes also has nine double points.

In fact, Segre did try to justify his claim, but in a totally unconvincing way. There is no discussion of the irreducibility of the relevant incidence correspondences. What is more, the argument depends on computing self-intersection numbers on a rather unspecified system of quartic surfaces, all of which might in particular be singular. However, this part of Segre's argument can be replaced by the following lemma. We always work over an algebraically closed ground field k of characteristic 0.

LEMMA 1.1. *No K3 surface carries any one-dimensional (non-constant) algebraic system of irreducible rational curves, whether smooth or singular.*

Proof. A smooth rational curve Γ on a K3 surface has arithmetic genus zero, and hence $(\Gamma)^2 = -2$. Therefore Γ is not even numerically equivalent to any other irreducible curve. This yields the assertion for smooth curves.

As for the singular case, a proof is sketched in [G-G] (Lemma 4.2), but we supply more details. Let B be a parameter curve for an algebraic system of curves on a K3 surface X, whose generic member is irreducible and rational. Without loss of generality, B is irreducible and even smooth, since we are free to replace it by its normalization. Let $\mathcal{J} \subset X \times B$ be the subvariety of codimension 1 corresponding to the algebraic system. Then, by [Sh] (Chap. 1, §6, Thm. 8), $\mathcal{J}$ is irreducible. We denote by $p: \mathcal{J} \to X$ the first projection and observe that p is dominant, unless the family is constant.

Let η be a generic point of B over the ground field k, so that $k(\eta) = k(B)$. By assumption, the fibre Γ_η of $\mathcal{J}$ above η is rational over the algebraic closure $\overline{k(\eta)}$ of $k(\eta)$. So, by a result which goes back to Hilbert and Hurwitz, Γ_η is birationally equivalent over $k(\eta)$ to a smooth conic. Now, k is algebraically closed; so, if t is a variable then $k(t)$ is a C_1 field (*cf.* [La], Thm. 6).

As the function field of some curve, $k(\eta)$ is an algebraic extension of $k(t)$; hence it is also C_1 ([La], pp. 376–377). So, every conic defined over $k(\eta)$ has points defined over this field and is birationally equivalent to $\mathbf{P}^1_{k(\eta)}$. This shows that $k(\eta)(\Gamma_\eta)$ is isomorphic to $k(\eta)(t)$. Therefore we have the following k-isomorphisms:

$$k(\mathcal{J}) \approx k(\eta)(\Gamma_\eta) \approx k(\eta)(t) = k(B \times \mathbf{P}^1).$$

Hence there is a birational equivalence $\varphi\colon B \times \mathbf{P}^1 \dashrightarrow \mathcal{J}$. Consider the composite rational map $q = p \circ \varphi\colon B \times \mathbf{P}^1 \dashrightarrow X$. Since q is dominant, and X projective, we know (cf. [Sh], Chap. 3, §5, Thm. 2) that q^* embeds the regular differentials (of any rank) on X into those on $B \times \mathbf{P}^1$.

Since X is a K3 surface, we note that ω_X is trivial, and hence $h^0(\omega_X) = 1$. On applying q^* we see that $h^0(B \times \mathbf{P}^1, \omega_{B \times \mathbf{P}^1}) \neq 0$. But this is impossible. Indeed, if we denote by p_1 and p_2 the projections from $B \times \mathbf{P}^1$ to B and $\mathbf{P}^1$ respectively, we have:

$$\omega_{B \times \mathbf{P}^1} = p_1^* \omega_B \otimes p_2^* \omega_{\mathbf{P}^1}.$$

On the other hand, $H^0(\mathbf{P}^1, \omega_{\mathbf{P}^1}) = H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2)) = 0$, and for quasi-coherent sheaves the global section functor commutes with tensor products; a contradiction. $\square$

REMARK.   Lemma 1.1 does not imply that a given K3 surface cannot contain infinitely many smooth rational curves; see [SwD], §5, for an example.

2. ABOUT EXISTENCE

Finiteness statements are useless if they are not accompanied by some form of existence assertion. After all, zero is also a finite number! In the present section we show the existence of irreducible rational curves, of degree 8 or 12, at least on some smooth quartic surfaces. For degree 8 there is a very elementary proof, and we give it first. Then we shall proceed to the case of degree 12, which requires some more elaborate machinery.

As mentioned above, on a quartic surface it is easy to find some reducible curves of degree 8 with nine double points by considering unions of two plane sections. Such curves are even infinite in number, but they do not lie on any smooth quadric.[4] That is why we start with a very explicit construction on

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[4]) By the way, this may be one reason for working with the Chow variety rather than with a Hilbert scheme. These degenerate cases have the same arithmetic genus, but they do not lie in $\mathcal{R}_8$.

the smooth quadric S which is the image of the standard Segre embedding

$$\sigma: \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$$

$$\big((x_0 : x_1), (y_0 : y_1)\big) \mapsto (X : Y : Z : W) = (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1).$$

Thus S is given by the equation

$$G(X, Y, Z, W) = XW - YZ = 0.$$

LEMMA 2.1. *Let* $\rho: \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ *be the map defined by*

$$\rho: (u : t) \mapsto \big((u^4 : t^4), (u^4 : u^3 t + t^4)\big).$$

Then ρ *is an injective morphism, whose image is an irreducible rational curve* Γ *of type* $(4, 4)$. *Under the standard Segre embedding,* Γ *is the intersection of* S *with the quartic surface* T *defined by*

$$F(X, Y, Z, W) = (Y - Z)^4 - X^3 Z = 0.$$

Of course, T *is a cone with vertex* $P = (0 : 0 : 0 : 1)$ *and has a triple line* $\ell = \{X = Y - Z = 0\}$. *However,* Γ *is also the intersection of* S *with a smooth quartic surface of the form* $F + H \cdot G = 0$ *for some quadratic form* $H(X, Y, Z, W)$.

Proof. All the assertions are easy to verify. Γ has a unique singularity (at P), whose effect on the genus is the equivalent of nine double points. As for the last assertion, we state it in more general form:

LEMMA 2.2. *Let* $\Gamma \subset \mathbf{P}^3$ *be the complete intersection of two surfaces defined by* $F = 0$, *respectively* $G = 0$. *We assume that the surface defined by* $G = 0$ *is smooth, that* Γ *is reduced, and that* $\deg F \geq \deg G$. *Then there exists a smooth surface among those with equation* $F + H \cdot G = 0$, *where* $\deg H = \deg F - \deg G$.

Proof. By a theorem of Bertini, the linear system determined by F and by all polynomials of the form $H \cdot G$ has no movable singularity in $\mathbf{P}^3$ outside its base locus. As H runs through the set of all forms of the relevant degree, the base locus is reduced to the points on $\Gamma = \{F = G = 0\}$.

Now, if P is a singular point of $F + H \cdot G = 0$ in the base locus, we see that $dF(P) + H(P) \cdot dG(P) = 0$. We can think of this as a system of four equations in one variable $x = H(P)$. But the rank of the Jacobian matrix $(F', G')_P$ at P is equal to 1 or 2 (0 is ruled out because the surface defined

by $G = 0$ is smooth). If it is equal to 2 then there is no suitable x; hence P is not singular for any H.

When the rank is equal to 1, there is a unique solution and we get one linear condition in the affine space of the coefficients of H. However, this occurs only at the finitely many singular points of Γ. Since a finite union of hyperplanes does not exhaust the space of parameters, we can choose H so that its coefficients lie outside this union. For any such H, the surface $F + H \cdot G = 0$ is smooth on the whole of Γ. $\square$

As a further illustration, we show how to produce an example with nine distinct double points.

LEMMA 2.3.   *Let $\rho \colon \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ be the map defined by*

$$\rho \colon (u : t) \mapsto \left((u^4 : u^4 + u^2 t^2 + t^4), (u^4 + u^3 t + u t^3 : u^4 + t^4) \right).$$

Then ρ is a generically injective morphism, whose image is an irreducible rational curve Γ of type $(4,4)$ with precisely 9 distinct ordinary double points. Under the standard Segre embedding, Γ is the intersection of S with a smooth quartic surface.

Proof.   In view of Lemma 2.2, the main thing to do is to study the singularities of ρ. To this effect, we note that a polynomial map $\rho_0 \colon \mathbf{A}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$, defined by

$$\rho_0 \colon t \mapsto \left((\varphi_0(t) : \varphi_1(t)), (\psi_0(t) : \psi_1(t)) \right),$$

fails to be injective when we have the following simultaneous equalities

$$\frac{\varphi_1(t)}{\varphi_0(t)} = \frac{\varphi_1(\tau)}{\varphi_0(\tau)} \quad \text{and} \quad \frac{\psi_1(t)}{\psi_0(t)} = \frac{\psi_1(\tau)}{\psi_0(\tau)}$$

for two different values t and τ. Therefore we define

$$\alpha(t) = \frac{\varphi_0(\tau)\varphi_1(t) - \varphi_1(\tau)\varphi_0(t)}{t - \tau} \in \mathbf{C}[\tau][t]$$

and

$$\beta(t) = \frac{\psi_0(\tau)\psi_1(t) - \psi_1(\tau)\psi_0(t)}{t - \tau} \in \mathbf{C}[\tau][t].$$

Then ρ_0 fails to be injective if, after fixing τ, there exists $t \neq \tau$ such that $\alpha(t) = \beta(t) = 0$. This involves studying the resultant $R(\tau)$ of $\alpha(t)$ and $\beta(t)$ over $\mathbf{C}[\tau]$. If ρ_0 is generically injective then $R(\tau)$ is not identically zero. With our assumptions, it is a polynomial of degree ≤ 18, whose roots describe

the 9 pairs of points that are mapped to the double points of Γ. In fact, the degree is equal to 18 if we work projectively and consider ρ instead of ρ_0.

In the present case we obtain $R(\tau) = (\tau^2 + 1)(\tau^4 + 1)g(\tau)$, where

$$g(\tau) = \tau^{12} + 3\tau^{10} + 8\tau^8 + 2\tau^7 + 11\tau^6 + 6\tau^5 + 9\tau^4 + 8\tau^3 + 6\tau^2 + 4\tau + 1 \, .$$

This is a decomposition into **Q**-irreducible factors; hence all the roots are distinct. Furthermore, the degree is equal to 18, which means that the image of the point at infinity is smooth. (For the example of Lemma 2.1, one obtains $R(\tau) = 1$, which means that the whole singularity is concentrated at the image of the point at infinity.) $\square$

The approach we have taken for these examples can also serve to prove some general statements:

LEMMA 2.4. *The rational, reduced and irreducible curves of bidegree* (μ, ν) *on a smooth quadric in* **P**^{3} *are parametrized by an irreducible quasi-projective variety* $\mathcal{R}_{\mu,\nu} \subset \mathcal{R}_m$ *of dimension* $2m - 1$, *where* $m = \mu + \nu$. *A general point on* $\mathcal{R}_{\mu,\nu}$ *corresponds to an irreducible curve whose only singularities are distinct nodes.*

Proof. Any smooth quadric surface is isomorphic to **P**$^1 \times$ **P**^{1}. Further, a rational irreducible curve of bidegree (μ, ν) on **P**$^1 \times$ **P**^{1} is the image of a map $\rho: $ **P**$^1 \to$ **P**$^1 \times$ **P**^{1}, where $\rho = \big((\varphi_0 : \varphi_1), (\psi_0 : \psi_1)\big)$ consists of two pairs of homogeneous polynomials, respectively of degree μ and ν, varying independently. These maps are parametrized by points of **P**$^{2\mu+1} \times$ **P**$^{2\nu+1}$.

This defines an incidence correspondence $\mathcal{T}$, with base the open subset V of **P**$^{2\mu+1} \times$ **P**$^{2\nu+1}$ which parametrizes those ρ which are generically injective and for which $\rho($**P**$^1)$ is of bidegree (μ, ν). Indeed, the condition that ρ be "many-to-one" is equivalent to the vanishing of some resultant polynomial (as in the proof of Lemma 2.3).

The argument given in [Co2], Lemma 2.4, shows that $\mathcal{T}$ is irreducible and that there is a correspondence between V and an irreducible subvariety $\mathcal{R}_{\mu,\nu}$ of $\mathcal{R}_m$ which defines the same curves as V. As the ∞^3 automorphisms of **P**^{1} do not modify the image of a map, the dimension of $\mathcal{R}_{\mu,\nu}$ is equal to $(2\mu + 1) + (2\nu + 1) - 3 = 2m - 1$, provided $\mathcal{R}_{\mu,\nu}$ is nonempty.

For $\mu = \nu = 4$ this is shown by Lemma 2.1, and the last assertion of the lemma follows from Lemma 2.3.

For the general case, we refer to [Ta], as in Lemma 2.5 below. More precisely, take $2 \leq \mu \leq \nu$ and assume by induction the existence of a nodal,

irreducible, rational curve Y_1 of bidegree $(\mu - 1, \nu)$. Let Y_2 be a line of type $(1, 0)$ avoiding the $(\mu - 2)(\nu - 1)$ nodes of Y_1. We can also assume that it meets Y_1 in exactly ν distinct points. Assign $\nu - 1$ of these, in addition to the nodes of Y_1. This set of $(\mu - 1)(\nu - 1)$ nodes makes $Y_1 + Y_2$ virtually connected, as desired. $\square$

We also know from Lemma 2.2 that any reduced curve of type (μ, μ) lies on a *smooth* surface of degree μ, a fact that will play an important role later.

It is difficult to pursue such an explicit approach for the case where $h = 3$, because the smooth cubic surfaces are not all alike. We therefore switch to the method of Tannenbaum [Ta]. His results, which are based on deformation theory, provide the existence – under precise conditions – of rational, *reduced and irreducible* curves, parametrized by an algebraic scheme. Unfortunately, they fail to apply on surfaces which are not rational. That is why we cannot immediately generalize our results to the intersections of a quartic with surfaces of degree higher than 3.

LEMMA 2.5. *Any smooth cubic surface $F \subset \mathbf{P}^3$ carries (for any positive integer λ) a rational, reduced and irreducible curve Γ_λ of degree 3λ having only nodes for singularities, which belongs to the linear system $|X_\lambda|$ cut out by all surfaces of degree λ.*

Proof. Since the surface F is rational, we can use the results of Tannenbaum ([Ta], §2). The proof is by induction on λ.

For $\lambda = 1$ we consider the intersection Γ_P of F with its tangent plane at any point P that does not lie on any of the 27 lines. Then Γ_P is irreducible. If P is sufficiently general then Γ_P has a node at P. Indeed, one also obtains a node with the plane sections of F that degenerate into the union of a line and a smooth conic. Thus there are many ways to choose Γ_P; we shall use this ∞^2-freedom in the rest of the proof.

Now suppose the result true for λ. We prove it for $\lambda + 1$. Thus we assume that there exists a rational, reduced and irreducible curve $\Gamma_\lambda \in |X_\lambda|$ with $p_a(\Gamma_\lambda) = 3\frac{\lambda(\lambda - 1)}{2} + 1$ distinct nodes, as the genus formula shows. (Indeed, F is embedded in $\mathbf{P}^3$ by its anticanonical sheaf.)

We apply [Ta], Prop. 2.11, to the reduced curve $Y = \Gamma_\lambda \cup \Gamma_P$, where Γ_P is a sufficiently general rational plane section. Then the 3λ intersection points of Γ_λ with Γ_P are among the nodes of Y, which therefore totals

$$\delta = 3\frac{\lambda(\lambda - 1)}{2} + 1 + 3\lambda + 1 = 3\frac{\lambda(\lambda + 1)}{2} + 2$$

nodes. Of course Y belongs to $|X_{\lambda+1}|$ and we may assign $\bar{\delta} = \delta - 1$ of the nodes, leaving out only one of the intersection points of Γ_λ with Γ_P.

In this way we obtain a flat family $\mathfrak{Y}$ relative to which these nodes are assigned; and Y is virtually connected with respect to $\mathfrak{Y}$, in the sense of [Ta], Def. 2.12. Thus we can apply [Ta], Thm. 2.13, to deduce that a generic member of $|X_{\lambda+1}|$ with $\bar{\delta} = 3\frac{\lambda(\lambda+1)}{2} + 1$ nodes is irreducible. By the genus formula, such a curve is rational. $\square$

With this existence result we can now state the analogue of Lemma 2.4 for cubics.

LEMMA 2.6. *The rational, reduced and irreducible curves, of degree $m = 3\lambda$, belonging to the linear system $|X_\lambda|$ on a smooth cubic surface $F \subset \mathbf{P}^3$ and having only nodes for singularities are parametrized by a quasi-projective, equidimensional scheme $\mathcal{W}_\lambda \subset \mathcal{R}_m$ of dimension $m - 1$.*

Proof. We apply [Ta], Lemma 2.2, to the rational, reduced and irreducible curve $Y = \Gamma_\lambda \in |X_\lambda|$ of the preceding lemma, which has precisely $\delta = p_a(Y) = 3\frac{\lambda(\lambda-1)}{2} + 1$ nodes and no other singular points. Further, $h^0(Y) = p_a(Y) + \deg Y$ (cf. [Co1], Lemma 1).

We derive the existence of a smooth irreducible algebraic k-scheme $V_\delta(|X_\lambda|; Y)$, of dimension $\dim |X_\lambda| - \delta = h^0(Y) - 1 - p_a(Y) = m - 1$, parametrizing reduced curves in $|X_\lambda|$ with precisely δ nodes and no other singularities which are flat deformations of Y in F. Of course, a general curve of $V_\delta(|X_\lambda|; Y)$ is irreducible.

Let $\mathfrak{Y} \subset V_\delta \times F$ be the universal Cartier divisor of the flat family. Since $V_\delta \times F$ is smooth, we can regard $\mathfrak{Y}$ as a Weil divisor, and hence as an incidence correspondence in this product. We prove that $\mathfrak{Y}$ is irreducible. Indeed, since $\mathfrak{Y} \xrightarrow{\varphi} V_\delta$ is flat, and every fibre is one-dimensional, it follows from [Hart], Chap. 3, Cor. 9.6, that every irreducible component $\mathfrak{Y}_i$ of $\mathfrak{Y}$ has dimension equal to $\dim V_\delta + 1$. Now, V_δ is irreducible, so $\overline{\varphi(\mathfrak{Y}_i)} = V_\delta$ for every i. But the generic fibre of φ is irreducible. Hence $i = 1$.

We denote by $\mathfrak{Y}'$ the open dense subset of $\mathfrak{Y}$ corresponding to the irreducible curves, and let $V'_\delta = \varphi(\mathfrak{Y}')$. We now apply [H-P], Chap. 11, §6, Thm. II, to $\mathfrak{Y}'$ and conclude that there is an irreducible incidence correspondence $\mathcal{T}$ between V'_δ and an irreducible subvariety $\mathcal{W}_Y \subset \mathcal{R}_m$, which defines the same curves as V'_δ.

Since every curve parametrized by V_δ is reduced, we have $\dim \mathcal{W}_Y = \dim V'_\delta$. Taking all possible irreducible $Y \in |X_\lambda|$, we get irreducible varieties

$\mathcal{W}_Y$ parametrizing all rational, reduced and irreducible curves belonging to $|X_\lambda|$ and having only nodes for singularities. We define $\mathcal{W}_\lambda$ as the union of these varieties $\mathcal{W}_Y$. $\square$

REMARKS. 1) The schemes parametrizing all curves of a given geometric genus, in a linear system on a rational surface, have been well examined (see [Ta], [Ha]). The new feature in Lemma 2.6 is that we "pull them up" to subschemes of the Chow variety $\mathcal{R}_m$.

2) We can think of a smooth cubic surface as being $\mathbf{P}^2$ with six points blown up. Then, if we consider the effect of blowing-down on the curves of the linear system $|X_\lambda|$, we see that Lemma 2.5 has the following interesting consequence: in the system of plane curves of degree 3λ with six λ-fold points, there are some rational, reduced and irreducible curves with only nodes as further singularities.

3. RATIONAL CURVES ON QUARTICS IN $\mathbf{P}^3$

A rational space curve of degree 8 is given as the image of a map $\varphi\colon \mathbf{P}^1 \to \mathbf{P}^3$, defined by four homogeneous polynomials of degree 8. Such maps depend on $4 \cdot 9 = 36$ arbitrary coefficients; hence they are parametrized by $\mathbf{P}^{35}$. Those maps which are generically injective and for which $\varphi(\mathbf{P}^1)$ is a curve of degree 8 correspond to an open subset $U \subset \mathbf{P}^{35}$. By $\varphi \in U$ we mean that the coefficients of φ are in U.

Such a curve Γ is contained in at most one quadric Q. So, it will be convenient to consider the pair (Γ, Q) instead of Γ alone. For simplicity, we shall restrict to the case where Q is smooth. We denote by $\mathcal{L}_Q$ (resp., $\mathcal{L}_C$ and $\mathcal{L}_K$) the quasi-projective variety of smooth quadrics (resp., cubics and quartics) in $\mathbf{P}^3$.

LEMMA 3.1. *The following correspondences between quasi-projective varieties are algebraic and define closed subvarieties:*

a) the incidence correspondence $\mathcal{G} \subset \mathcal{R}_8 \times \mathcal{L}_Q$ *parametrizing the rational curves of degree 8 on smooth quadrics;*

b) the incidence correspondence $\mathcal{F} \subset \mathcal{R}_8 \times \mathcal{L}_Q \times \mathcal{L}_K$ *parametrizing the rational, reduced and irreducible curves of type* $(4,4)$ *on smooth quadrics which are cut out by smooth quartic surfaces;*

c) the incidence correspondence $\mathcal{H} \subset \mathcal{R}_8 \times \mathcal{L}_Q \times \mathcal{F}_K$ *parametrizing the rational, reduced and irreducible curves of type* $(4,4)$ *on smooth quadrics.*

Proof. Let

$$F = \left\{ (\varphi, x, Q, K) \in U \times \mathbf{P}^3 \times \mathcal{L}_Q \times \mathcal{L}_K \mid x \in \varphi(\mathbf{P}^1) \subset Q \cap K \right\}.$$

This is a closed subset of $U \times \mathbf{P}^3 \times \mathcal{L}_Q \times \mathcal{L}_K$. Indeed, $\varphi(\mathbf{P}^1) \subset Q \cap K$ means that $\varphi(u : t) \in Q \cap K$ for every $(u : t) \in \mathbf{P}^1$. So, the coefficients of all monomials in $(u : t)$ of $f_Q(\varphi(u : t))$ and $f_K(\varphi(u : t))$ must vanish (here f_Q and f_K denote the polynomial equations of Q and K). This yields algebraic relations between x and the coefficients of φ, f_Q, and f_K.

Let F_i be any irreducible component of F. Call U_i its first projection. By [H-P] (Chap. 11, §6, Thm. II), there exists an irreducible correspondence $\mathcal{F}_i$ between $\mathcal{L}_Q \times \mathcal{L}_K$ and an irreducible subvariety $\mathcal{R}_{U_i}$ of $\mathcal{R}_8$ which defines the same curves as U_i. We define $\mathcal{F}$ to be the union of the $\mathcal{F}_i$.

Similarly, we define $G = \left\{ (\varphi, x, Q) \in U \times \mathbf{P}^3 \times \mathcal{L}_Q \mid x \in \varphi(\mathbf{P}^1) \subset Q \right\}$. Further, let $H = \left\{ (\varphi, x, Q, K) \in U \times \mathbf{P}^3 \times \mathcal{L}_Q \times \mathcal{F}_K \mid x \in \varphi(\mathbf{P}^1) \subset Q \cap K \right\}$. As before, G and H are closed subsets of $U \times \mathbf{P}^3 \times \mathcal{L}_Q$, respectively $U \times \mathbf{P}^3 \times \mathcal{L}_Q \times \mathcal{F}_K$. Making use of irreducible components G_i of G, respectively H_i of H, we find irreducible correspondences $\mathcal{G}_i$ and $\mathcal{H}_i$ between $\mathcal{L}_Q$, respectively $\mathcal{L}_Q \times \mathcal{F}_K$ and irreducible varieties $\mathcal{R}_{W_i} \subset \mathcal{R}_8$, respectively $\mathcal{R}_{V_i} \subset \mathcal{R}_8$. Again, we define $\mathcal{G}$ and $\mathcal{H}$ as the unions of these irreducible components.

Finally, we note that $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$, as closed subsets of quasi-projective varieties, are quasi-projective. $\square$

LEMMA 3.2. *The incidence correspondence $\mathcal{F}$ is a quasi-projective variety of dimension* 34.

Proof. Define $\pi_0 : \mathcal{G} \to \mathcal{L}_Q$ by $\pi_0(\gamma, Q) = Q$ (for $\gamma \in \mathcal{R}_8$) and similarly

$\pi_1 : \mathcal{H} \to \mathcal{G}$ by $\pi_1(\gamma, Q, K) = (\gamma, Q)$ and $\pi : \mathcal{F} \to \mathcal{G}$ by $\pi(\gamma, Q, K) = (\gamma, Q)$.

Now, $\mathcal{H}$ is a closed subset of a quasi-projective variety, and $\mathcal{F}_K$ is projective. It follows from [Sh] (Chap. 1, §5, Thm. 3) that $\pi_1(\mathcal{H})$ is closed in $\mathcal{G}$, and hence also quasi-projective.

Since a smooth quadric Q is projectively normal and a linear equivalence class on Q is determined by the bidegree, every reduced and irreducible curve Γ of bidegree $(4, 4)$ is cut out, on Q, by (at least) one irreducible quartic K. Let γ be the Chow point of Γ, so that $(\gamma, Q, K) \in \mathcal{H}$. Then the fibre of π_1 above (γ, Q) contains K and all the reducible quartics through Q. Hence it is of dimension 10.

We also know from Lemma 2.2 that a general member of the linear system of all quartics through Γ is smooth. Hence $\pi_1(\mathcal{H}) = \pi(\mathcal{F})$ and the fibres of π over $\pi(\mathcal{F})$ are also 10-dimensional.

Finally, by Lemma 2.4, $\pi_0 \circ \pi_1$ maps onto $\mathcal{L}_Q$ and all the fibres of $\pi_0|_{\pi(\mathcal{F})}$ have dimension 15. As $\mathcal{L}_Q$ is irreducible of dimension 9, any irreducible component of $\pi(\mathcal{F})$ of maximal dimension has dimension 24. Further, the fibre of π over $\pi(\mathcal{F})$ is 10-dimensional, so $\mathcal{F}$ has dimension 34. $\square$

LEMMA 3.3.   *The incidence correspondence $\mathcal{J} \subset \mathcal{R}_{12} \times \mathcal{L}_C \times \mathcal{L}_K$ parametrizing the rational, reduced and irreducible curves which are the complete intersection of a smooth quartic and a smooth cubic surface, is a quasi-projective variety of dimension 34.*

Proof.   The argument is much the same as for degree 8. For instance, a curve Γ of degree 12 cannot be contained in more than one cubic C. So, it is convenient to consider the pair (Γ, C) instead of Γ alone. For simplicity, we restrict to the case where C is smooth. Then Lemma 2.6 replaces Lemma 2.4, and the proof of Lemma 3.2 has to be modified mainly for the actual computation of dimensions, which is as follows.

The quasi-projective variety $\mathcal{L}_C$ of smooth cubic surfaces has dimension 19. And the dimension of the family of rational curves $\Gamma \in |X_4|$ is equal to 11, by[5]) Lemma 2.6. Finally, a curve $\Gamma \subset C$ lying on an irreducible quartic K, is also contained in all the reducible quartics through C. Hence the linear system of all quartics through Γ has dimension 4, and a general member is smooth.

Putting everything together, we find $19 + 11 + 4 = 34$ for the dimension of the incidence correspondence $\mathcal{J}$. $\square$

We now come to the proof of assertion (**S**) for $h = 2$ and $h = 3$.

THEOREM 3.4.   *The smooth quartics in $\mathbf{P}^3$ carrying rational, reduced and irreducible curves of degree 8, respectively 12, obtained as intersections with smooth quadrics, respectively cubics, form a constructible set of dimension 34 in $\mathcal{L}_K$. A general quartic in this set carries also some rational curves of degree 8 (resp., 12) having only nodes for singularities.*

Proof.   Let $p \colon \mathcal{F} \to \mathcal{L}_K$ be the projection map defined by $p(\gamma, Q, K) = K$. Then every fibre of p is finite. Indeed, let us consider the first projection

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[5]) Strictly speaking, this is only a lower bound, since Lemma 2.6 does not count the curves having other singularities than nodes. But the proof of Theorem 3.4 shows that one has in fact equality.

$q\colon \mathcal{F} \to \mathcal{R}_8$. We know from Lemma 3.1 that $\mathcal{F}$ is algebraic. Hence, for any $K \in \mathcal{L}_K$, the push-forward $q_* p^{-1}(K)$ describes an algebraic system of rational curves on K, which cannot be of dimension ≥ 1, by Lemma 1.1, since $K \in \mathcal{L}_K$ is a K3 surface. Hence this algebraic system is finite. Moreover, $\mathcal{F}$ parametrizes only reduced curves of degree 8, which therefore do not belong to more than one quadric. Hence each Chow point γ corresponds to a unique pair (γ, Q). Thus the fibre $p^{-1}(K)$ contains only finitely many points.

Let $E \subset \mathcal{F}$ be an irreducible component of top dimension 34. By the theorem on the dimension of the fibres (see [Sh], Chap. 1, §6, Thm. 7), we see that $\dim p(E) = \dim E = 34$.

On applying Chevalley's theorem to the quasi-projective varieties $\mathcal{F}$ and $\mathcal{L}_K$ and to the finite-type morphism p, we also see that $p(\mathcal{F})$ is constructible, i.e., a finite disjoint union of locally closed subsets V_i. Since $\mathcal{L}_K$ is quasi-projective, so are the V_i.

The same argument works for curves of degree 12, with the map $p\colon \mathcal{J} \to \mathcal{L}_K$. Note that one gets, for a component $E \subset \mathcal{J}$ of maximal dimension,

$$34 \geq \dim p(E) = \dim E \geq 34.$$

Hence we obtain the same equality as before. $\square$

REMARKS. 1) As expected, singular points other than nodes do not affect the dimensions of the relevant schemes. This is because, roughly speaking, nodes impose the lowest number of conditions for decreasing the geometric genus. However, as is shown by Lemma 2.1, not all curves in Theorem 3.4 have only nodes for singularities.

2) In the proof of Theorem 3.4, we could replace E by its closure $\bar{E}$ in $\mathcal{R}_8 \times \mathcal{F}_Q \times \mathcal{L}_K$, where $\mathcal{F}_Q$ denotes the space of all quadrics in $\mathbf{P}^3$. Now, $\mathcal{R}_8 \times \mathcal{F}_Q$ is a projective variety. Hence $p(\bar{E})$ is closed (cf. [Sh], Chap. 1, §5, Thm. 3) and $p(\bar{E}) = \mathcal{L}_K$. This would account in particular for the rational octics that lie on a quadratic cone, instead of a smooth quadric surface.

4. RATIONAL CURVES ON K3 SURFACES IN $\mathbf{P}^4$

Let $S_{2,3}$ be a K3 surface spanning $\mathbf{P}^4$ (i.e., not contained in any hyperplane). The notation refers to the fact that such a surface is a smooth complete intersection of a quadric and a cubic threefold. We also write $\mathcal{S}_{2,3}$ for the 43-dimensional quasi-projective variety of all $S_{2,3}$'s in $\mathbf{P}^4$ (see Lemma 4.2). In the present section we prove:

THEOREM 4.1.   *The surfaces in $S_{2,3}$ carrying rational integral curves of degree* 12, *obtained as intersections with smooth quadrics, form a constructible set of dimension* 43 *in* $S_{2,3}$.

The idea is to consider the curves at issue as belonging to the intersection of two quadrics in $\mathbf{P}^4$. This is a Del Pezzo surface (i.e., its anticanonical sheaf is ample). Hence it is not very different from a cubic surface. In particular, it is rational and we can apply again the results of Tannenbaum.

We write $\mathcal{P}_4$ for the quasi-projective variety of all smooth intersections of two quadrics in $\mathbf{P}^4$. Thus, $\mathcal{P}_4$ is an open subset of the Grassmann variety of pencils of quadrics in $\mathbf{P}^4$.

LEMMA 4.2. $\mathcal{P}_4$ *has dimension* 26; *and* $S_{2,3}$ *has dimension* 43.

Proof.   The dimension of $\mathcal{P}_4$ is the dimension of the Grassmann variety of rank 2 subspaces of the space of quadratic forms in 5 variables, to wit, $2(15-2) = 26$.

Similarly, $S_{2,3}$ is an open subset of the projective bundle over the space $\mathbf{P}^{14}$ of quadrics with fibre the projectivization of the space of cubic forms modulo (linear) multiples of a quadric. Thus the fibre has dimension $\binom{3+4}{4} - 5 - 1 = 29$.   □

LEMMA 4.3.   *Any smooth intersection of two quadrics* $P \subset \mathbf{P}^4$ *carries* (*for any positive integer* λ) *a rational, reduced and irreducible curve* Γ_λ *of degree* $m = 4\lambda$ *having only nodes for singularities, which belongs to the linear system* $|X_\lambda|$ *cut out by all hypersurfaces of degree* λ.

Such curves are parametrized by an irreducible quasi-projective scheme of dimension $m-1$.

Proof.   We simply note that $P \in \mathcal{P}_4$ is embedded in $\mathbf{P}^4$ by its anticanonical sheaf. Hence we can apply [Co1], Lemma 1, and the proofs of Lemma 2.5 and Lemma 2.6 carry over with minimal changes.   □

In the present paper we are especially interested in the case where $\lambda = 3$. The lemma shows that there exist rational, integral curves of degree $m = 12$ on some surfaces $S \in S_{2,3}$. They are obtained as intersections with smooth quadrics and have only nodes for singularities.

Proof of Theorem 4.1.   Let $\mathcal{G}_{12}$ be the Chow variety of rational curves of degree 12 in $\mathbf{P}^4$. As explained at the beginning of §3, we can work over an

open set of reduced and irreducible curves. This will be implied whenever we write a new correspondence. (For simplicity we shall use the same notation, Γ, for a curve and for its Chow point.)

We denote by $\mathcal{L}_Q$ (resp., $\mathcal{L}_C$) the quasi-projective varieties of smooth quadric (resp., cubic) threefolds in $\mathbf{P}^4$. As in Lemma 3.1, the incidence correspondences we are working with can be "pulled up" to define the following algebraic correspondences:

$$\mathcal{H} = \{(\Gamma, P) \in \mathcal{G}_{12} \times \mathcal{P}_4 \mid \Gamma = P \cap C \text{ for some } C \in \mathcal{L}_C \}$$

and

$$\mathcal{J} = \{(\Gamma, S) \in \mathcal{G}_{12} \times \mathcal{S}_{2,3} \mid \Gamma = S \cap Q \text{ for some } Q \in \mathcal{L}_Q \}.$$

In view of Lemmas 4.2 and 4.3, the dimension of $\mathcal{H}$ is equal to $26 + (m-1) = 37$. This is also the dimension of its image in $\mathcal{G}_{12}$. Indeed, the fibres of the second projection are finite, since a curve $\Gamma \in \mathcal{G}_{12}$ cannot belong to more than one intersection of two quadrics. (In fact, there is even a map that goes directly from $\mathcal{J}$ to $\mathcal{H}$, but we can do without it.)

To compute the dimension of $\mathcal{J}$, we notice that $\mathcal{H}$ and $\mathcal{J}$ have the same image in $\mathcal{G}_{12}$. Now, a general element in the image of $\mathcal{H}$ corresponds to a curve Γ of degree 12 with 13 distinct nodes and belongs to a pencil of quadrics. But a surface $S \in \mathcal{S}_{2,3}$ is contained in a unique quadric. Hence an element in the fibre of $\mathcal{J}$ above Γ determines a quadric, say Q, in the ∞^1-system of quadrics through Γ and is then determined by the family of all cubic threefolds containing Γ, provided we discount the reducible elements that contain Q.

On the other hand, no more than 24 conditions are required for a cubic hypersurface to contain Γ. In fact it is enough to impose 11 simple points and the 13 double points, since this represents $2 \cdot 13 + 11 = 3 \cdot 12 + 1$ intersections. Therefore, as a vector space, the family of cubics containing Γ has dimension $(\geq) \; 35 - 24 = 11$.

After discounting, as in Lemma 4.2, the 5-dimensional vector space of reducible cubics containing Q as a component, we are left with an ∞^5-system of surfaces[6]) $S \in \mathcal{S}_{2,3}$ containing Γ and contained in Q. As Q varies in a pencil, the fibre of $\mathcal{J}$ above Γ has dimension $(\geq) \; 5 + 1 = 6$.

It follows that $\mathcal{J}$ is of dimension $(\geq) \; 37 + 6 = 43$. Now, let p be the projection map from $\mathcal{J}$ to $\mathcal{S}_{2,3}$. By Lemma 1.1 all the fibres of this map are finite. Since the dimensions are right, as is shown by Lemma 4.2, we conclude exactly as in the proof of Theorem 3.4. $\square$

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[6]) The smoothness can be proved by an extension of Lemma 2.2, in which we replace the divisors in $\mathbf{P}^3$ by divisors in Q.

REMARK. Theorems 3.4 and 4.1, together with [C-S], Example 1.3, clearly imply the following statements:

THEOREM 3.4' *The smooth quartics in* $\mathbf{P}^3$ *carrying reduced and irreducible curves of degree* 8, *respectively* 12, *and geometric genus* $9 - \delta$ $(0 \leq \delta \leq 9)$, *respectively* $19 - \delta$ $(0 \leq \delta \leq 19)$, *obtained as intersections with smooth quadrics, respectively cubics, and having* δ *nodes, form a constructible set of dimension* 34 *in* $\mathcal{L}_K$.

THEOREM 4.1' *The surfaces in* $\mathcal{S}_{2,3}$ *carrying integral curves of degree* 12 *and geometric genus* $13 - \delta$ $(0 \leq \delta \leq 13)$, *obtained as intersections with smooth quadrics, form a constructible set of dimension* 43 *in* $\mathcal{S}_{2,3}$.

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