2. About existence

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As the function field of some curve, $k(\eta)$ is an algebraic extension of $k(t)$; hence it is also C_1 ([La], pp. 376–377). So, every conic defined over $k(\eta)$ has points defined over this field and is birationally equivalent to $\mathbf{P}^1_{k(\eta)}$. This shows that $k(\eta)(\Gamma_\eta)$ is isomorphic to $k(\eta)(t)$. Therefore we have the following k-isomorphisms:

$$k(\mathcal{J}) \approx k(\eta)(\Gamma_\eta) \approx k(\eta)(t) = k(B \times \mathbf{P}^1).$$

Hence there is a birational equivalence $\varphi\colon B \times \mathbf{P}^1 \dashrightarrow \mathcal{J}$. Consider the composite rational map $q = p \circ \varphi\colon B \times \mathbf{P}^1 \dashrightarrow X$. Since q is dominant, and X projective, we know (*cf.* [Sh], Chap. 3, §5, Thm. 2) that q^* embeds the regular differentials (of any rank) on X into those on $B \times \mathbf{P}^1$.

Since X is a K3 surface, we note that ω_X is trivial, and hence $h^0(\omega_X) = 1$. On applying q^* we see that $h^0(B \times \mathbf{P}^1, \omega_{B\times\mathbf{P}^1}) \neq 0$. But this is impossible. Indeed, if we denote by p_1 and p_2 the projections from $B \times \mathbf{P}^1$ to B and $\mathbf{P}^1$ respectively, we have:

$$\omega_{B\times\mathbf{P}^1} = p_1^*\omega_B \otimes p_2^*\omega_{\mathbf{P}^1}.$$

On the other hand, $H^0(\mathbf{P}^1, \omega_{\mathbf{P}^1}) = H^0\big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(-2)\big) = 0$, and for quasi-coherent sheaves the global section functor commutes with tensor products; a contradiction. $\square$

REMARK. Lemma 1.1 does not imply that a given K3 surface cannot contain infinitely many smooth rational curves; see [SwD], §5, for an example.

2. ABOUT EXISTENCE

Finiteness statements are useless if they are not accompanied by some form of existence assertion. After all, zero is also a finite number! In the present section we show the existence of irreducible rational curves, of degree 8 or 12, at least on some smooth quartic surfaces. For degree 8 there is a very elementary proof, and we give it first. Then we shall proceed to the case of degree 12, which requires some more elaborate machinery.

As mentioned above, on a quartic surface it is easy to find some reducible curves of degree 8 with nine double points by considering unions of two plane sections. Such curves are even infinite in number, but they do not lie on any smooth quadric.[4]) That is why we start with a very explicit construction on

[4]) By the way, this may be one reason for working with the Chow variety rather than with a Hilbert scheme. These degenerate cases have the same arithmetic genus, but they do not lie in $\mathcal{R}_8$.

the smooth quadric S which is the image of the standard Segre embedding

$$\sigma\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$$
$$\big((x_0 : x_1), (y_0 : y_1)\big) \mapsto (X : Y : Z : W) = (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1).$$

Thus S is given by the equation

$$G(X, Y, Z, W) = XW - YZ = 0.$$

LEMMA 2.1. *Let $\rho\colon \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ be the map defined by*

$$\rho\colon (u : t) \mapsto \big((u^4 : t^4), (u^4 : u^3t + t^4)\big).$$

Then ρ is an injective morphism, whose image is an irreducible rational curve Γ of type $(4, 4)$. Under the standard Segre embedding, Γ is the intersection of S with the quartic surface T defined by

$$F(X, Y, Z, W) = (Y - Z)^4 - X^3Z = 0.$$

Of course, T is a cone with vertex $P = (0 : 0 : 0 : 1)$ and has a triple line $\ell = \{X = Y - Z = 0\}$. However, Γ is also the intersection of S with a smooth quartic surface of the form $F + H \cdot G = 0$ for some quadratic form $H(X, Y, Z, W)$.

Proof. All the assertions are easy to verify. Γ has a unique singularity (at P), whose effect on the genus is the equivalent of nine double points. As for the last assertion, we state it in more general form:

LEMMA 2.2. *Let $\Gamma \subset \mathbf{P}^3$ be the complete intersection of two surfaces defined by $F = 0$, respectively $G = 0$. We assume that the surface defined by $G = 0$ is smooth, that Γ is reduced, and that* $\deg F \geq \deg G$. *Then there exists a smooth surface among those with equation $F + H \cdot G = 0$, where* $\deg H = \deg F - \deg G$.

Proof. By a theorem of Bertini, the linear system determined by F and by all polynomials of the form $H \cdot G$ has no movable singularity in $\mathbf{P}^3$ outside its base locus. As H runs through the set of all forms of the relevant degree, the base locus is reduced to the points on $\Gamma = \{F = G = 0\}$.

Now, if P is a singular point of $F + H \cdot G = 0$ in the base locus, we see that $dF(P) + H(P) \cdot dG(P) = 0$. We can think of this as a system of four equations in one variable $x = H(P)$. But the rank of the Jacobian matrix $(F', G')_P$ at P is equal to 1 or 2 (0 is ruled out because the surface defined by $G=0$ is smooth). If it is equal to 2 then there is no suitable x; hence P is not singular for any H.

When the rank is equal to 1, there is a unique solution and we get one linear condition in the affine space of the coefficients of H. However, this occurs only at the finitely many singular points of Γ. Since a finite union of hyperplanes does not exhaust the space of parameters, we can choose H so that its coefficients lie outside this union. For any such H, the surface $F+H\cdot G=0$ is smooth on the whole of Γ. □

As a further illustration, we show how to produce an example with nine distinct double points.

LEMMA 2.3. *Let $\rho\colon \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$ be the map defined by*

$$\rho\colon (u:t) \mapsto \big((u^4 : u^4+u^2t^2+t^4), (u^4+u^3t+ut^3 : u^4+t^4)\big)\,.$$

Then ρ is a generically injective morphism, whose image is an irreducible rational curve Γ of type $(4,4)$ with precisely 9 distinct ordinary double points. Under the standard Segre embedding, Γ is the intersection of S with a smooth quartic surface.

Proof. In view of Lemma 2.2, the main thing to do is to study the singularities of ρ. To this effect, we note that a polynomial map $\rho_0\colon \mathbf{A}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$, defined by

$$\rho_0\colon t \mapsto \big((\varphi_0(t):\varphi_1(t)), (\psi_0(t):\psi_1(t))\big)\,,$$

fails to be injective when we have the following simultaneous equalities

$$\frac{\varphi_1(t)}{\varphi_0(t)} = \frac{\varphi_1(\tau)}{\varphi_0(\tau)} \quad \text{and} \quad \frac{\psi_1(t)}{\psi_0(t)} = \frac{\psi_1(\tau)}{\psi_0(\tau)}$$

for two different values t and τ. Therefore we define

$$\alpha(t) = \frac{\varphi_0(\tau)\varphi_1(t) - \varphi_1(\tau)\varphi_0(t)}{t-\tau} \in \mathbf{C}[\tau][t]$$

and

$$\beta(t) = \frac{\psi_0(\tau)\psi_1(t) - \psi_1(\tau)\psi_0(t)}{t-\tau} \in \mathbf{C}[\tau][t]\,.$$

Then ρ_0 fails to be injective if, after fixing τ, there exists $t \neq \tau$ such that $\alpha(t)=\beta(t)=0$. This involves studying the resultant $R(\tau)$ of $\alpha(t)$ and $\beta(t)$ over $\mathbf{C}[\tau]$. If ρ_0 is generically injective then $R(\tau)$ is not identically zero. With our assumptions, it is a polynomial of degree ≤ 18, whose roots describe the 9 pairs of points that are mapped to the double points of Γ. In fact, the degree is equal to 18 if we work projectively and consider ρ instead of ρ_0.

In the present case we obtain $R(\tau) = (\tau^2 + 1)(\tau^4 + 1)g(\tau)$, where

$$g(\tau) = \tau^{12} + 3\tau^{10} + 8\tau^8 + 2\tau^7 + 11\tau^6 + 6\tau^5 + 9\tau^4 + 8\tau^3 + 6\tau^2 + 4\tau + 1 .$$

This is a decomposition into $\mathbf{Q}$-irreducible factors; hence all the roots are distinct. Furthermore, the degree is equal to 18, which means that the image of the point at infinity is smooth. (For the example of Lemma 2.1, one obtains $R(\tau) = 1$, which means that the whole singularity is concentrated at the image of the point at infinity.) $\square$

The approach we have taken for these examples can also serve to prove some general statements:

LEMMA 2.4. *The rational, reduced and irreducible curves of bidegree (μ, ν) on a smooth quadric in $\mathbf{P}^3$ are parametrized by an irreducible quasi-projective variety $\mathcal{R}_{\mu,\nu} \subset \mathcal{R}_m$ of dimension $2m - 1$, where $m = \mu + \nu$. A general point on $\mathcal{R}_{\mu,\nu}$ corresponds to an irreducible curve whose only singularities are distinct nodes.*

Proof. Any smooth quadric surface is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Further, a rational irreducible curve of bidegree (μ, ν) on $\mathbf{P}^1 \times \mathbf{P}^1$ is the image of a map $\rho\colon \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$, where $\rho = \big((\varphi_0 : \varphi_1), (\psi_0 : \psi_1)\big)$ consists of two pairs of homogeneous polynomials, respectively of degree μ and ν, varying independently. These maps are parametrized by points of $\mathbf{P}^{2\mu+1} \times \mathbf{P}^{2\nu+1}$.

This defines an incidence correspondence $\mathcal{T}$, with base the open subset V of $\mathbf{P}^{2\mu+1} \times \mathbf{P}^{2\nu+1}$ which parametrizes those ρ which are generically injective and for which $\rho(\mathbf{P}^1)$ is of bidegree (μ, ν). Indeed, the condition that ρ be "many-to-one" is equivalent to the vanishing of some resultant polynomial (as in the proof of Lemma 2.3).

The argument given in [Co2], Lemma 2.4, shows that $\mathcal{T}$ is irreducible and that there is a correspondence between V and an irreducible subvariety $\mathcal{R}_{\mu,\nu}$ of $\mathcal{R}_m$ which defines the same curves as V. As the ∞^3 automorphisms of $\mathbf{P}^1$ do not modify the image of a map, the dimension of $\mathcal{R}_{\mu,\nu}$ is equal to $(2\mu + 1) + (2\nu + 1) - 3 = 2m - 1$, provided $\mathcal{R}_{\mu,\nu}$ is nonempty.

For $\mu = \nu = 4$ this is shown by Lemma 2.1, and the last assertion of the lemma follows from Lemma 2.3.

For the general case, we refer to [Ta], as in Lemma 2.5 below. More precisely, take $2 \le \mu \le \nu$ and assume by induction the existence of a nodal, irreducible, rational curve Y_1 of bidegree $(\mu-1,\nu)$. Let Y_2 be a line of type $(1,0)$ avoiding the $(\mu-2)(\nu-1)$ nodes of Y_1. We can also assume that it meets Y_1 in exactly ν distinct points. Assign $\nu-1$ of these, in addition to the nodes of Y_1. This set of $(\mu-1)(\nu-1)$ nodes makes Y_1+Y_2 virtually connected, as desired. □

We also know from Lemma 2.2 that any reduced curve of type (μ,μ) lies on a *smooth* surface of degree μ, a fact that will play an important role later.

It is difficult to pursue such an explicit approach for the case where $h=3$, because the smooth cubic surfaces are not all alike. We therefore switch to the method of Tannenbaum [Ta]. His results, which are based on deformation theory, provide the existence – under precise conditions – of rational, *reduced and irreducible* curves, parametrized by an algebraic scheme. Unfortunately, they fail to apply on surfaces which are not rational. That is why we cannot immediately generalize our results to the intersections of a quartic with surfaces of degree higher than 3.

LEMMA 2.5. *Any smooth cubic surface $F\subset \mathbf{P}^3$ carries (for any positive integer λ) a rational, reduced and irreducible curve Γ_λ of degree 3λ having only nodes for singularities, which belongs to the linear system $|X_\lambda|$ cut out by all surfaces of degree λ.*

Proof. Since the surface F is rational, we can use the results of Tannenbaum ([Ta], §2). The proof is by induction on λ.

For $\lambda=1$ we consider the intersection Γ_P of F with its tangent plane at any point P that does not lie on any of the 27 lines. Then Γ_P is irreducible. If P is sufficiently general then Γ_P has a node at P. Indeed, one also obtains a node with the plane sections of F that degenerate into the union of a line and a smooth conic. Thus there are many ways to choose Γ_P; we shall use this ∞^2-freedom in the rest of the proof.

Now suppose the result true for λ. We prove it for $\lambda+1$. Thus we assume that there exists a rational, reduced and irreducible curve $\Gamma_\lambda\in|X_\lambda|$ with $p_a(\Gamma_\lambda)=3\frac{\lambda(\lambda-1)}{2}+1$ distinct nodes, as the genus formula shows. (Indeed, F is embedded in $\mathbf{P}^3$ by its anticanonical sheaf.)

We apply [Ta], Prop. 2.11, to the reduced curve $Y=\Gamma_\lambda\cup\Gamma_P$, where Γ_P is a sufficiently general rational plane section. Then the 3λ intersection points of Γ_λ with Γ_P are among the nodes of Y, which therefore totals

$$\delta=3\frac{\lambda(\lambda-1)}{2}+1+3\lambda+1=3\frac{\lambda(\lambda+1)}{2}+2$$

nodes. Of course Y belongs to $|X_{\lambda+1}|$ and we may assign $\bar{\delta} = \delta - 1$ of the nodes, leaving out only one of the intersection points of Γ_λ with Γ_P.

In this way we obtain a flat family $\mathfrak{Y}$ relative to which these nodes are assigned; and Y is virtually connected with respect to $\mathfrak{Y}$, in the sense of [Ta], Def. 2.12. Thus we can apply [Ta], Thm. 2.13, to deduce that a generic member of $|X_{\lambda+1}|$ with $\bar{\delta} = 3\frac{\lambda(\lambda+1)}{2} + 1$ nodes is irreducible. By the genus formula, such a curve is rational. $\square$

With this existence result we can now state the analogue of Lemma 2.4 for cubics.

LEMMA 2.6. *The rational, reduced and irreducible curves, of degree $m = 3\lambda$, belonging to the linear system $|X_\lambda|$ on a smooth cubic surface $F \subset \mathbf{P}^3$ and having only nodes for singularities are parametrized by a quasi-projective, equidimensional scheme $\mathcal{W}_\lambda \subset \mathcal{R}_m$ of dimension $m-1$.*

Proof. We apply [Ta], Lemma 2.2, to the rational, reduced and irreducible curve $Y = \Gamma_\lambda \in |X_\lambda|$ of the preceding lemma, which has precisely $\delta = p_a(Y) = 3\frac{\lambda(\lambda-1)}{2} + 1$ nodes and no other singular points. Further, $h^0(Y) = p_a(Y) + \deg Y$ (cf. [Co1], Lemma 1).

We derive the existence of a smooth irreducible algebraic k-scheme $V_\delta(|X_\lambda|; Y)$, of dimension $\dim|X_\lambda| - \delta = h^0(Y) - 1 - p_a(Y) = m - 1$, parametrizing reduced curves in $|X_\lambda|$ with precisely δ nodes and no other singularities which are flat deformations of Y in F. Of course, a general curve of $V_\delta(|X_\lambda|; Y)$ is irreducible.

Let $\mathfrak{Y} \subset V_\delta \times F$ be the universal Cartier divisor of the flat family. Since $V_\delta \times F$ is smooth, we can regard $\mathfrak{Y}$ as a Weil divisor, and hence as an incidence correspondence in this product. We prove that $\mathfrak{Y}$ is irreducible. Indeed, since $\mathfrak{Y} \xrightarrow{\varphi} V_\delta$ is flat, and every fibre is one-dimensional, it follows from [Hart], Chap. 3, Cor. 9.6, that every irreducible component $\mathfrak{Y}_i$ of $\mathfrak{Y}$ has dimension equal to $\dim V_\delta + 1$. Now, V_δ is irreducible, so $\overline{\varphi(\mathfrak{Y}_i)} = V_\delta$ for every i. But the generic fibre of φ is irreducible. Hence $i = 1$.

We denote by $\mathfrak{Y}'$ the open dense subset of $\mathfrak{Y}$ corresponding to the irreducible curves, and let $V'_\delta = \varphi(\mathfrak{Y}')$. We now apply [H-P], Chap. 11, §6, Thm. II, to $\mathfrak{Y}'$ and conclude that there is an irreducible incidence correspondence $\mathcal{T}$ between V'_δ and an irreducible subvariety $\mathcal{W}_Y \subset \mathcal{R}_m$, which defines the same curves as V'_δ.

Since every curve parametrized by V_δ is reduced, we have $\dim \mathcal{W}_Y = \dim V'_\delta$. Taking all possible irreducible $Y \in |X_\lambda|$, we get irreducible varieties

$\mathcal{W}_Y$ parametrizing all rational, reduced and irreducible curves belonging to $|X_\lambda|$ and having only nodes for singularities. We define $\mathcal{W}_\lambda$ as the union of these varieties $\mathcal{W}_Y$. □

REMARKS. 1) The schemes parametrizing all curves of a given geometric genus, in a linear system on a rational surface, have been well examined (see [Ta], [Ha]). The new feature in Lemma 2.6 is that we "pull them up" to subschemes of the Chow variety $\mathcal{R}_m$.

2) We can think of a smooth cubic surface as being $\mathbf{P}^2$ with six points blown up. Then, if we consider the effect of blowing-down on the curves of the linear system $|X_\lambda|$, we see that Lemma 2.5 has the following interesting consequence: in the system of plane curves of degree 3λ with six λ-fold points, there are some rational, reduced and irreducible curves with only nodes as further singularities.

3. RATIONAL CURVES ON QUARTICS IN $\mathbf{P}^3$

A rational space curve of degree 8 is given as the image of a map $\varphi \colon \mathbf{P}^1 \to \mathbf{P}^3$, defined by four homogeneous polynomials of degree 8. Such maps depend on $4 \cdot 9 = 36$ arbitrary coefficients; hence they are parametrized by $\mathbf{P}^{35}$. Those maps which are generically injective and for which $\varphi(\mathbf{P}^1)$ is a curve of degree 8 correspond to an open subset $U \subset \mathbf{P}^{35}$. By $\varphi \in U$ we mean that the coefficients of φ are in U.

Such a curve Γ is contained in at most one quadric Q. So, it will be convenient to consider the pair (Γ, Q) instead of Γ alone. For simplicity, we shall restrict to the case where Q is smooth. We denote by $\mathcal{L}_Q$ (resp., $\mathcal{L}_C$ and $\mathcal{L}_K$) the quasi-projective variety of smooth quadrics (resp., cubics and quartics) in $\mathbf{P}^3$.

LEMMA 3.1. *The following correspondences between quasi-projective varieties are algebraic and define closed subvarieties:*

a) the incidence correspondence $\mathcal{G} \subset \mathcal{R}_8 \times \mathcal{L}_Q$ parametrizing the rational curves of degree 8 on smooth quadrics;

b) the incidence correspondence $\mathcal{F} \subset \mathcal{R}_8 \times \mathcal{L}_Q \times \mathcal{L}_K$ parametrizing the rational, reduced and irreducible curves of type $(4,4)$ on smooth quadrics which are cut out by smooth quartic surfaces;

c) the incidence correspondence $\mathcal{H} \subset \mathcal{R}_8 \times \mathcal{L}_Q \times \mathcal{F}_K$ parametrizing the rational, reduced and irreducible curves of type $(4,4)$ on smooth quadrics.