

# EVEN NON-SPIN MANIFOLDS, SPIN<sub>c</sub> STRUCTURES, AND DUALITY

Autor(en): **ACOSTA, Daniel / LAWSON, Terry**

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## EVEN NON-SPIN MANIFOLDS, SPIN<sup>c</sup> STRUCTURES, AND DUALITY

by Daniel ACOSTA and Terry LAWSON

**ABSTRACT.** This note explores the restrictions on the second Stiefel-Whitney class  $w_2$  of a smooth closed oriented 4-manifold which has an even intersection form but is not spin. The Hom dual of  $w_2$  is shown to be non-integral, whereas the existence of a spin<sup>c</sup> structure means that its Poincaré dual is the reduction of an integral class. We examine this in detail in a simple example  $S^2 \times S^2 / \{\pm 1\}$ .

In [H, p. 23] N. Habegger gives  $M = S^2 \times S^2 / (x, y) \sim (-x, -y)$  as an example of an oriented, non-spin smooth 4-manifold with an even intersection form. In discussing this example in [K, p. 27] R. Kirby seems to be relating it to the (non-)integrality of the second Stiefel-Whitney class  $w_2(M)$ . However, for a closed, oriented, smooth 4-manifold  $X$ , it is always the case that the second Stiefel-Whitney class  $w_2(X)$  is the mod 2 reduction of an integral class. This was first shown by Hirzebruch and Hopf in [HH, p. 169], and is a key step in showing that  $X$  admits a spin<sup>c</sup> structure. Spin<sup>c</sup> structures have recently become very important as they are involved in the Seiberg-Witten invariants, now a major area of study in the topology and geometry of 4-manifolds (see, e.g., [W], [T], [KM]).

In this expository paper we want to explore some of the interesting phenomena at work in this example and describe the characterizing property which  $w_2$  possesses. On the way we shall encounter many important concepts in geometric topology, including Poincaré and Hom duality, the intersection form, even forms, spin structures, spin<sup>c</sup> structures, and characteristic classes. The article is intended for readers who have a background of a year-long graduate course in topology from a text such as Bredon [B].

We start by reviewing some basic definitions. We will assume  $X$  is a compact, oriented smooth 4-manifold. When the coefficient group is not

written, it is assumed to be the integers. There are two forms of duality available which we will use. First, Poincaré duality asserts that cap product with the fundamental class  $[X] \in H_4(X)$  gives an isomorphism  $D : H^2(X) \rightarrow H_2(X)$ . There is a similar isomorphism when we use  $\mathbf{Z}_2$  coefficients which we will also denote by  $D$ . For coefficient group  $\mathbf{Z}_2$  there is an isomorphism  $H : H^2(X; \mathbf{Z}_2) \rightarrow \text{Hom}_{\mathbf{Z}_2}(H_2(X; \mathbf{Z}_2), \mathbf{Z}_2)$  with image the dual space of the vector space  $H_2(X; \mathbf{Z}_2)$  over the field  $\mathbf{Z}_2$ . A basis  $b_1, \dots, b_n$  of a finite dimensional vector space  $V$  determines an isomorphism between  $V$  and its dual  $V^*$  by sending  $b_i$  to the homomorphism  $B_i$  which sends  $b_i$  to 1 and  $b_j$  to 0 for  $j \neq i$ . The elements  $B_i$  and  $b_i$  are said to be *Hom duals*. This isomorphism depends on a choice of basis. However, if we are given any elements  $b \in V$ ,  $B \in V^*$ , with  $B(b) = 1$ , then we can always extend  $b = b_1$  to a basis of  $V$  so that  $b$  is the Hom dual of  $B$  — just extend  $b$  to any basis and then subtract off appropriate multiples of  $b$  to get  $B$  to evaluate 0 on the other basis elements. The composition of the isomorphism  $H$  and the isomorphism determined by the basis gives an isomorphism

$$\bar{H} : H^2(X; \mathbf{Z}_2) \simeq \text{Hom}_{\mathbf{Z}_2}(H_2(X; \mathbf{Z}_2), \mathbf{Z}_2) \simeq H_2(X; \mathbf{Z}_2)$$

which will be called *Hom duality*. We will call  $x \in H_2(X; \mathbf{Z}_2)$  a *Hom dual* of  $h \in H^2(X; \mathbf{Z}_2)$  if  $H(h)(x) = 1$  since we can always choose a basis of  $H_2(X; \mathbf{Z}_2)$  so that  $\bar{H}(h) = x$ .

We next explore briefly the notions of an even intersection form, spin structure, and  $\text{spin}^c$  structure for a compact, oriented smooth 4-manifold  $X$ . For more details see [B, p. 366–378], [K, p. 20–26, 33–37], [A, p. 95–101], [M, p. 20–25]. The intersection form  $H_2(X) \times H_2(X) \rightarrow \mathbf{Z}$  is defined by using the intersection product  $a \cdot b$  of two homology classes. If the homology classes are represented by smoothly embedded oriented surfaces  $A, B$  (i.e. the inclusion maps induce  $(i_A)_*[A] = a, (i_B)_*[B] = b$ ), then  $a \cdot b$  may be computed by perturbing  $A, B$  up to isotopy to be transversely embedded and summing up the intersections with signs  $\pm 1$  according to whether the orientation framing of  $A$  followed by the orientation framing of  $B$  agrees or disagrees with the orientation framing of  $X$  [B, p. 375]. It is always the case that a 2-dimensional homology class in an oriented 4-manifold may be represented by an embedded surface [K, p. 20]. The product  $a \cdot b$  may also be computed using Poincaré duality as  $a \cdot b = \alpha \cup \beta[X] = \alpha(b)$ , where  $D\alpha = a, D\beta = b$ . There are similar formulas with  $\mathbf{Z}_2$  coefficients. A two dimensional  $\mathbf{Z}_2$  homology class is not always represented by an embedded oriented surface, but it always may be represented by an embedded nonorientable surface [G, p. 165–166], and there is a similar interpretation of the intersection form in terms of counting

geometric transverse intersections. The map  $H_2(X) \rightarrow H_2(X; \mathbf{Z}_2)$  is surjective exactly when every  $\mathbf{Z}_2$  homology class can be represented by an orientable surface.

The universal coefficient sequences with integral and  $\mathbf{Z}_2$  coefficients lead to the following diagram.

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \text{Ext}(H_1(X), \mathbf{Z}) & \longrightarrow & H^2(X; \mathbf{Z}) & \xrightarrow{h_1} & \text{Hom}(H_2(X), \mathbf{Z}) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \\
 0 & \longrightarrow & \text{Ext}(H_1(X), \mathbf{Z}_2) & \longrightarrow & H^2(X; \mathbf{Z}_2) & \xrightarrow{h_2} & \text{Hom}(H_2(X), \mathbf{Z}_2) & \longrightarrow & 0
 \end{array}$$

The homomorphisms  $h_1$  and  $h_2$  are related to the intersection form:

$$h_1(\alpha)(b) = a \cdot b, \quad h_2(\alpha)(b) = a \cdot b \pmod{2}$$

where  $D(\alpha) = a$  with either  $\mathbf{Z}$  or  $\mathbf{Z}_2$  coefficients. The homomorphisms  $\rho_i$  come from reduction mod 2. The intersection form is called *even* if  $x \cdot x$  is an even number for all  $x \in H_2(X)$ . An integral class  $a \in H_2(X)$  so that  $a \cdot x = x \cdot x \pmod{2}$  for all  $x$  is called *characteristic* for the intersection form.  $a$  is characteristic if the homomorphism  $S(x) = x \cdot x \pmod{2}$  is the image of  $a$  under the homomorphism  $k: H_2(X; \mathbf{Z}) \rightarrow \text{Hom}(H_2(X), \mathbf{Z}_2)$  where  $k(a)(x) = a \cdot x \pmod{2}$ . If  $a$  is a characteristic class, and  $\alpha$  is its Poincaré dual, then  $h_1(\alpha)(x) = a \cdot x = x \cdot x \pmod{2}$ . Thus  $a$  is characteristic iff its Poincaré dual  $\alpha$  satisfies  $h_2 \rho_1(\alpha) = \rho_2 h_1(\alpha) = S$ . Since the form is even iff  $S = 0$ , this means that the form is even iff for  $a$  characteristic,  $D\alpha = a$ , then  $h_2(\rho_1(\alpha)) = 0$ .

The existence of characteristic classes uses the nondegeneracy of the intersection form and Poincaré duality with  $\mathbf{Z}_2$  coefficients. The intersection pairing  $H_2(X; \mathbf{Z}) \times H_2(X; \mathbf{Z}) \rightarrow \mathbf{Z}$  factors through  $\Gamma \times \Gamma \rightarrow \mathbf{Z}$  where  $\Gamma = H_2(X; \mathbf{Z})/\text{Tors}$ , and when we reduce mod 2, through  $\Gamma_2 \times \Gamma_2 \rightarrow \mathbf{Z}_2$  where  $\Gamma_2 = \Gamma \otimes \mathbf{Z}_2$ . The existence follows from  $\Gamma \rightarrow \Gamma_2$  being surjective and  $\Gamma_2 \rightarrow \text{Hom}(\Gamma_2, \mathbf{Z}_2)$  being an isomorphism. For this last isomorphism, note both sides are  $\mathbf{Z}_2$ -vector spaces and have dimension equal to  $\text{rank } H_2(X; \mathbf{Z})$ . The isomorphism is established once the map is seen to be injective. This follows from the fact that the intersection form is nondegenerate due to Poincaré duality: for each  $v, \exists w$  with  $w \cdot v = 1$ ; in fact,  $w = D\psi$  where  $\psi$  is the Hom dual of  $v$ :

$$w \cdot v = D\psi \cdot v = H(\psi)(v) = 1.$$

The second Stiefel-Whitney class  $w_2(X) \in H^2(X; \mathbf{Z}_2)$  belongs to a family of characteristic classes. A good reference for properties of the Stiefel-Whitney

classes and characteristic classes in general is [MS]. For our discussion here we need to know three of its properties. First, it is related to the characteristic classes discussed above in that its Poincaré dual  $D(w_2(X))$  satisfies the characteristic property for the  $\mathbf{Z}_2$  intersection form:

$$H(w_2(X))(z) = D(w_2(X)) \cdot z = z \cdot z$$

for all  $z \in H_2(X; \mathbf{Z}_2)$ . When we restrict to the image of integral classes, we get the statement that  $h_2(w_2(X))(x) = x \cdot x \pmod{2}$ . This means that if  $D(\alpha_1)$  is an integral characteristic class, then  $h_2(w_2 - \rho_1(\alpha_1)) = 0$ . The second property that  $w_2(X)$  satisfies is that an oriented manifold  $X$  has a spin structure iff  $w_2(X) = 0$ . A spin structure on  $X$  is a lifting of the structure group of the tangent bundle of  $X$  from  $SO(4)$  to its universal (double)cover  $spin(4)$ . The third property which  $w_2(X)$  possesses relates to  $spin^c$  structures. The group  $spin^c(4)$  is the double cover  $spin(4) \times S^1 / \pm 1$  of  $SO(4) \times S^1$  induced from the double cover on each factor. A  $spin^c$  structure on  $X$  consists of a lifting of the structure group of the product of the tangent bundle of  $X$  and a chosen line bundle  $L$  over  $X$  from  $SO(4) \times S^1$  to  $spin^c(4)$ . The 4-manifold  $X$  has a  $spin^c$  structure exactly when the second Stiefel-Whitney class  $w_2(X) = \rho_1(\alpha)$  for some integral class  $\alpha$  ([HH, p. 169], [M, p. 25]).

We now give the argument why  $w_2(X)$  always lifts to an integral class from the excellent expository account of Seiberg-Witten invariants by S. Akbulut [A, p. 95]. We saw above that the existence of an integral characteristic class means there is an integral class  $\alpha_1$  so that  $h_2(w_2(X) - \rho_1(\alpha_1)) = 0$ . Hence  $w_2 - \rho_1(\alpha_1)$  comes from  $\text{Ext}(H_1(X), \mathbf{Z}_2)$ . But the map  $\text{Ext}(H_1(X), \mathbf{Z}) \rightarrow \text{Ext}(H_1(X), \mathbf{Z}_2)$  is surjective since the first group gives the torsion subgroup of  $H_1(X)$  and the latter the 2-torsion subgroup. Hence  $\exists \alpha_2 \in \text{Ext}(H_1(X), \mathbf{Z}) \hookrightarrow H^2(X; \mathbf{Z})$  with  $\rho_1(\alpha_2) = w_2 - \rho_1(\alpha_1)$ . This implies  $w_2 = \rho_1(\alpha_1 + \alpha_2)$  is the image of an integral cohomology class. Note that this also means that the Poincaré dual  $D(w_2)$  is the image of an integral homology class.

With this background, we return now to our initial example  $M$ . To see that  $w_2(M) \neq 0$ , Habegger [H] notes that if  $\mathbf{RP}^2 = \{[(x, x)]\}$  is the image of the diagonal  $\Delta$  in  $S^2 \times S^2$  under the quotient, then  $[\Delta] \cdot [\Delta] = 2$  in  $S^2 \times S^2$  leads to  $[\mathbf{RP}^2] \cdot [\mathbf{RP}^2] = 1$  in  $M$ . If  $[\mathbf{RP}^2] = D\gamma$ , where  $\gamma \in H^2(M; \mathbf{Z}_2)$ , then we have  $(\gamma \cup \gamma)[M] = [\mathbf{RP}^2] \cdot [\mathbf{RP}^2] = 1$ . Thus  $w_2(M) \cup \gamma = \gamma \cup \gamma \neq 0$ , which implies  $w_2(M) \neq 0$  and thus  $M$  is not spin.

Next note  $\pi_1(M) = \mathbf{Z}_2 = H_1(M)$  since  $M$  is double covered by  $S^2 \times S^2$ . Using this and the computation of Euler characteristic as  $\chi(M) = \chi(S^2 \times S^2)/2 = 2$ , Habegger shows  $\text{rank } H_2(M) = 0$ . Evenness of the

intersection form follows. The universal coefficient sequences for  $M$  are:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \mathbf{Z}_2 & \xrightarrow{\cong} & \mathbf{Z}_2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 \downarrow & & \downarrow \cong & & \downarrow \rho & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \longrightarrow & \mathbf{Z}_2 & \longrightarrow & 0
 \end{array}$$

Consider the homology class  $Dw_2$ . We claim that it is represented by the embedded sphere which is the image under the quotient of  $S^2 \times p$  or  $p \times S^2$  in  $S^2 \times S^2$ . Here  $p$  is a chosen point in  $S^2$ , say  $(1,0,0)$ . To see this, note that  $(S^2 \times p) \cap \Delta = (p,p)$  and the intersection is transverse. This gives us  $[S^2 \times p]_2 \cdot [\mathbf{RP}^2] = 1$  in  $M$ , and  $[S^2 \times p]_2$  is therefore a nonzero class in  $H_2(M; \mathbf{Z}_2)$  — the subscript 2 indicates that here we are viewing  $[S^2 \times p]$  as a  $\mathbf{Z}_2$  homology class rather than an integral class. This implies  $[S^2 \times p]$  must be nonzero in  $H_2(M) \simeq \mathbf{Z}_2$ . Its Poincaré dual in  $H^2(M) \simeq \mathbf{Z}_2$  must therefore be the unique nonzero class which reduces mod 2 to  $w_2(M)$ . This is reflected in our commutative diagram. Evenness is reflected through the upper right term being zero, and the image of  $w_2$  to the Hom term being zero. Exactness implies  $w_2 \in \mathbf{Z}_2 \oplus \mathbf{Z}_2$  must come from the Ext term. Note that under the isomorphism  $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \simeq H^2(M; \mathbf{Z}_2) \simeq \text{Hom}_{\mathbf{Z}_2}(H_2(M; \mathbf{Z}_2), \mathbf{Z}_2)$ ,  $w_2$  maps to a nonzero homomorphism which evaluates zero on  $[S^2 \times p]_2$  and one on  $[\mathbf{RP}^2]$ .

What is true here is that the class  $[\mathbf{RP}^2]$  in  $H_2(M; \mathbf{Z}_2)$  does not come from an integral class. The evaluation of  $w_2$  on  $[\mathbf{RP}^2]$  and  $[S^2 \times p]_2$  distinguishes these classes. Thus, these two surfaces generate  $H_2(M; \mathbf{Z}_2) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$  and the intersection form with respect to this basis is just  $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ . We also note that  $[\mathbf{RP}^2]$  cannot be represented by an oriented surface  $N$ . If it were,  $[N]$  would represent an element of  $H_2(M)$ , and as we have seen,  $[\mathbf{RP}^2]$  is not in the image of the homomorphism  $H_2(M) \longrightarrow H_2(M; \mathbf{Z}_2)$  since the form is even.

How typical is this example? First, if  $X$  has an even intersection form and  $w_2(X) \neq 0$ , then there must be a class  $a \in H_2(X; \mathbf{Z}_2)$  with  $a \cdot a \neq 0$  detecting  $w_2(X) \neq 0$  so that  $a$  does not come from an integral class. This class  $a$  can be taken as a Hom dual of  $w_2(X)$ , not the Poincaré dual. In our example,  $[\mathbf{RP}^2]$  is the Hom dual to  $w_2(M)$  (using the basis  $[S^2 \times p]_2, \mathbf{RP}^2$  to form the duality) since  $H(w_2(M))([\mathbf{RP}^2]) = 1$  and  $H(w_2(M))([S^2 \times p]_2) = 0$ . Of course, no such example can have  $H_2(X) \longrightarrow H_2(X; \mathbf{Z}_2)$  surjective, which implies  $X$  is not simply connected. Secondly,  $H_2(X; \mathbf{Z}_2)$  is always represented by embedded surfaces, orientable or nonorientable. All classes in the image

$H_2(X) \longrightarrow H_2(X; \mathbf{Z}_2)$  are represented by orientable surfaces. Thus, for  $X$  even, non-spin, some classes in  $H_2(X; \mathbf{Z}_2)$  will have only non-orientable representatives, such as  $[\mathbf{RP}^2]$  in our example.

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Daniel Acosta

Department of Mathematics  
Marshall University  
Huntington, WV 25755  
U.S.A.  
*E-mail*: acosta@marshall.edu

Terry Lawson

Department of Mathematics  
Tulane University  
New Orleans, LA 70118  
U.S.A.  
*E-mail*: tcl@math.tulane.edu