COROLLARY 3. *Let* λ *be a partition of* k *and* s_λ *the corresponding Schur polynomial in* $\Lambda(n, \mathbf{Q})$. *Then* $\widetilde{s_\lambda}(\overline{\mathcal{E}}) = 0$ *unless* λ *is a hook* $\lambda^i = (i, 1, 1, \ldots, 1)$, *in which case* $\widetilde{s_{\lambda^i}}(\overline{\mathcal{E}}) = (-1)^{k-i}\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$.

Proof. The proof is based on the Frobenius formula

$$s_\lambda = \frac{1}{k!}\sum_{\sigma\in S_k}\chi_\lambda(\sigma)p_{(\sigma)}$$

where (σ) denotes the partition of k determined by the cycle structure of σ (cf. [M], §I.7). By the above remark, $\widetilde{s_\lambda}(\overline{\mathcal{E}}) = \chi_\lambda\big((12\ldots k)\big)\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$. Using the combinatorial rule for computing χ_λ found in [M], p. 117, Example 5, we obtain

$$\chi_\lambda((12\ldots k)) = \begin{cases} (-1)^{k-i}, & \text{if } \lambda = \lambda^i \text{ is a hook} \\ 0, & \text{otherwise}. \end{cases}$$ □

The most natural instance of a sequence $\overline{\mathcal{E}}$ with $\overline{E}$ flat is the classifying sequence over the Grassmannian $G(r, n)$. As we shall see in §8, the calculation of Bott-Chern forms for this sequence leads to a presentation of the Arakelov Chow ring of the arithmetic Grassmannian over $\operatorname{Spec}\mathbf{Z}$.

6. CALCULATIONS WHEN $\overline{E}$ IS PROJECTIVELY FLAT

We will now generalize the results of the last section to the case where E is *projectively flat*, i.e. the curvature matrix K_E of $\overline{E}$ is a multiple of the identity matrix: $K_E = \omega\, Id_n$. This is true if $E = \overline{L}^{\oplus n}$ for some hermitian line bundle $\overline{L}$, with $\omega = c_1(\overline{L})$ the first Chern form of $\overline{L}$.

The Bott-Chern forms (for the induced metrics) are always closed in this case as well, and will be expressed in terms of characteristic classes of the bundles involved. However this seems to be the most general case where this phenomenon occurs.

The key observation is that for projectively flat bundles, the curvature matrix $K_E = \omega Id_n$ in *any* local trivialization. Thus we have

$$K(u) = \left(\begin{array}{c|c} (1-u)K_S + u\,\omega\, Id_r & 0 \\ \hline 0 & (1-u)K_Q + u\,\omega\, Id_s \end{array}\right)$$

where $s = n - r$ denotes the rank of Q. Now Theorem 2 gives

$$\widetilde{p_k}(\mathcal{E}) = k\int_0^1 \frac{1}{u}\,\mathrm{Tr}\left[\left(u\omega Id_r + (1-u)K_S\right)^{k-1} - K_S^{k-1}\right] du =$$

$$-k\mathcal{H}_{k-1}p_{k-1}(\bar{S}) + k\sum_{j=1}^{k-1}\binom{k-1}{j}\mathrm{Tr}(\omega^j K_S^{k-1-j})\int_0^1 u^{j-1}(1-u)^{k-j-1}\,du\,.$$

Integrating by parts gives $\displaystyle\int_0^1 u^m(1-u)^n\,du = \frac{1}{m+n+1}\binom{m+n}{n}^{-1}$, thus

$$\frac{1}{k}\widetilde{p_k}(\bar{\mathcal{E}}) = -\mathcal{H}_{k-1}p_{k-1}(\bar{S}) + \sum_{j=1}^{k-1}\frac{\omega^j}{j}p_{k-1-j}(\bar{S})\,. \tag{6}$$

We can rewrite this as an equation involving power sums of the quotient bundle: since $p_k(\bar{S}) + p_k(\bar{Q}) - p_k(\bar{L}^{\oplus n}) = 0$ in $\widetilde{A}(X)$, we have $p_k(\bar{S}) = n\omega^k - p_k(\bar{Q})$. Thus (6) becomes

$$\frac{1}{k}\widetilde{p_k}(\bar{\mathcal{E}}) = \mathcal{H}_{k-1}p_{k-1}(\bar{Q}) - \sum_{j=1}^{k-1}\frac{\omega^j}{j}p_{k-1-j}(\bar{Q})\,. \tag{7}$$

THEOREM 4. *Let X be a complex manifold, $\bar{E}$ a projectively flat hermitian vector bundle over X. Let $0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0$ a short exact sequence of vector bundles over X with metrics on S, Q induced from $\bar{E}$. Then for any invariant polynomial $\phi \in I(n)$, $\phi(\bar{S}\bigoplus\bar{Q}) = \phi(\bar{E})$ as differential forms on X.*

Proof. Since the p_λ form an additive basis for $I(n)$, it suffices to prove the result when $\phi = p_\lambda$. The above calculation shows that $\widetilde{p_k}$ is a closed form. This combined with Proposition 1 shows that $\widetilde{p_\lambda}$ is closed for any partition λ. Thus

$$p_\lambda(\bar{S}\bigoplus\bar{Q}) - p_\lambda(\bar{E}) = dd^c\widetilde{p_\lambda} = 0\,. \qquad \square$$

REMARK. If E is a trivial vector bundle, this result follows by pulling back the exact sequence $\bar{\mathcal{E}}$ from the classifying sequence on the Grassmannian. The forms are equal there because they are invariant with respect to the $U(n)$ action, so harmonic.

The Bott-Chern forms $\widetilde{p_\lambda}$ for a general partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ can be computed by using Proposition 1. If $|\lambda| = \sum\lambda_i = k$ then we have

$$\widetilde{p_\lambda}(\bar{\mathcal{E}}) = \sum_{i=1}^m \widetilde{p_{\lambda_i}}(\bar{\mathcal{E}})\prod_{j\neq i}p_{\lambda_j}(\bar{E}) = n^{m-1}\sum_{i=1}^m \omega^{k-\lambda_i}\widetilde{p_{\lambda_i}}(\bar{\mathcal{E}})\,. \tag{8}$$

In principle equations (7) and (8) can be used to compute $\widetilde{\phi}(\overline{\mathcal{E}})$ for any characteristic class ϕ.

We now find a more explicit formula for the Bott-Chern forms of Chern classes. The computation is not as straightforward, as the argument of Proposition 3 does not apply. Since by Theorem 2 the calculation depends only on the curvature matrices K_E, K_S and K_Q, we may assume

$$\overline{\mathcal{E}}\ :\ 0 \to \bar{S} \to \bar{L} \otimes \mathbf{C}^n \to \bar{Q} \to 0$$

is our chosen sequence, and define a new sequence

$$\overline{\mathcal{E}'} = \overline{\mathcal{E}} \otimes \bar{L}^*\ :\ 0 \to \bar{S} \otimes \bar{L}^* \to \mathbf{C}^n \to \bar{Q} \otimes \bar{L}^* \to 0\,.$$

The metrics on the bundles in $\overline{\mathcal{E}'}$ are induced from the trivial metric on $\mathbf{C}^n$. Using Propositions 2 and 3 now gives

$$\begin{aligned}
\widetilde{c_k}(\overline{\mathcal{E}}) = \widetilde{c_k}(\overline{\mathcal{E}'} \otimes \bar{L}) &= \sum_{i=1}^{k} \binom{n-i}{k-i} \widetilde{c_i}(\overline{\mathcal{E}'}) c_1(\bar{L})^{k-i} \\
&= \sum_{i=1}^{k} \binom{n-i}{k-i} (-1)^{i-1} \mathcal{H}_{i-1} p_{i-1}(\bar{Q} \otimes \bar{L}^*) \omega^{k-i} \\
&= \sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j \binom{n-i}{k-i} \binom{i-1}{j} \mathcal{H}_{i-1} \omega^{k-1-j} p_j(\bar{Q}) \\
&= \sum_{j=0}^{k-1} (-1)^j d_j \omega^{k-1-j} p_j(\bar{Q}),
\end{aligned}$$

where

$$d_j = \sum_{i=j+1}^{k} \binom{n-i}{k-i} \binom{i-1}{j} \mathcal{H}_{i-1}\,.$$

To find a closed form for the sum d_j, we can use the general identity

$$\text{(9)} \qquad \sum_{i=q-s}^{n-p} \binom{n-i}{p} \binom{s+i}{q} \mathcal{H}_{s+i} = \binom{n+s+1}{p+q+1} (\mathcal{H}_{n+s+1} - \mathcal{H}_{p+q+1} + \mathcal{H}_p)\,.$$

This is identity (10) in [Sp]. In passing we note that writing equation (9) without the harmonic number terms:

$$\sum_{i=q-s}^{n-p} \binom{n-i}{p} \binom{s+i}{q} = \binom{n+s+1}{p+q+1}$$

gives a well known identity among binomial coefficients. Applying (9) to d_j and replacing k by $k+1$ and j by $k-i$ we arrive at the formula

$$\widetilde{c_{k+1}}(\overline{\mathcal{E}}) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n}{i} \mathcal{H}_i \omega^i p_{k-i}(\overline{Q}),$$

where $\mathcal{H}_i = \mathcal{H}_n - \mathcal{H}_{n-i} + \mathcal{H}_{k-i}$. As remarked previously, this calculation is valid for any projectively flat bundle $\overline{E}$ with $c_1(\overline{E}) = n\omega$.

Of course one can use the above method to compute the Bott-Chern form $\widetilde{p_k}(\overline{\mathcal{E}})$ as well; however this leads to a more complicated formula than (7). Equating the two proves(!) the following interesting combinatorial identity (compare [Sp], identity (30)):

$$\text{(10)} \qquad \sum_{i=0}^{s} (-1)^{i+1} \binom{n}{i, s-i} \mathcal{H}_{n-s+i} = \frac{1}{s} \qquad (n \geqslant s).$$

Here $\binom{n}{i,j}$ is a trinomial coefficient.

The following summarizes the calculations of this section:

THEOREM 5. *Let X be a complex manifold, $\overline{E}$ a projectively flat hermitian vector bundle over X, with $c_1(\overline{E}) = n\omega$. Let $\overline{\mathcal{E}} : 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$ be a short exact sequence of vector bundles over X with metrics on S, Q induced from $\overline{E}$. Then*

$$\widetilde{p_{k+1}}(\overline{\mathcal{E}}) = (k+1)\mathcal{H}_k p_k(\overline{Q}) - (k+1) \sum_{i=1}^{k} \frac{\omega^i}{i} p_{k-i}(\overline{Q}),$$

$$\widetilde{c_{k+1}}(\overline{\mathcal{E}}) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n}{i} \mathcal{H}_i \omega^i p_{k-i}(\overline{Q}),$$

where $\mathcal{H}_i = \mathcal{H}_n - \mathcal{H}_{n-i} + \mathcal{H}_{k-i}$.

Note that the formulas in Theorem 5 reduce to the ones of the previous section when $\omega = 0$!