# 3. Similarities Normalise

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(ii) $S = \{F \in M_d(\mathbf{Q}) \mid F = F^{tr},\ F \text{ positive definite}\}$, the set of positive definite symmetric matrices, where $x^{tr}$ denotes the transposed matrix of $x \in M_d(\mathbf{Q})$ and the action of $H$ on $S$ is $S \times H \to S,\ (F,h) \mapsto hFh^{tr}$. Then the set of $G$-fixed points is

$$\mathcal{F}_{>0}(G) := \{F \in S \mid gFg^{tr} = F \text{ for all } g \in G\}.$$

Note that $(\mathbf{R}_{>0})\mathcal{F}_{>0}(G)$ is the set of $G$-invariant Euclidean scalar products on $\mathbf{R}^d$. $G$ is called *uniform*, if there is essentially one $G$-invariant Euclidean structure on $\mathbf{R}^d$, that is if $\mathcal{F}_{>0}(G) = \{aF \mid 0 < a \in \mathbf{Q}\}$ for some $F \in M_d(\mathbf{Q})$.

(iii) $S = M_d(\mathbf{Q})$, and the action of $H$ is conjugation: $S \times H \to S,\ (c,h) \mapsto h^{-1}ch$. Then the set of $G$-fixed points is the *commuting algebra* of $G$

$$C_{M_d(\mathbf{Q})}(G) := \{c \in M_d(\mathbf{Q}) \mid cg = gc \text{ for all } g \in G\}.$$

The following two remarks follow immediately from the normaliser principle.

REMARK 1. *Assume that $G$ is uniform and let $F \in \mathcal{F}_{>0}(G)$. Then for each $n \in N$, the matrix $nFn^{tr}$ is also $G$-invariant and therefore $nFn^{tr} = \left(\det(n)\right)^{2/d}F$. Hence $n$ induces a similarity of $F$.*

REMARK 2. *For $n \in N$ and $L \in \mathcal{Z}(G)$, the lattice $Ln \in \mathcal{Z}(G)$ is also $G$-invariant.*

## 3. SIMILARITIES NORMALISE

In this section we show that if $G$ is the automorphism group of a (strongly modular) lattice $L$ then the similarities between $L$ and $L' \in \pi(L)$ are elements of $N$.

PROPOSITION 3. *Let $G = \mathrm{Aut}(F) \leq GL_d(\mathbf{Z})$ be the full automorphism group of a lattice $L$. Assume that $L$ is an integral lattice. Let $L' \in \pi(L)$ and $n \in GL_d(\mathbf{Q})$ which induces a similarity from $L'$ to $L$, i.e. $L'n = L$ and $nFn^{tr} = aF$, ($a \in \mathbf{N}$). Then $a^{-1}n^2 \in G$ and $n \in N$.*

*Proof.* The matrix $a^{-1}n^2$ is clearly orthogonal with respect to $F$. Therefore to prove that $a^{-1}n^2 \in G$ we only have to show that $La^{-1}n^2 = L$. Now $L' = Ln^{-1}$, hence its dual lattice is

$$(L')^{\#} = \{v \in \mathbf{Q}^d \mid vF(ln^{-1})^{tr} \in \mathbf{Z} \quad \text{for all} \quad l \in L\}.$$

For $l \in L, v \in \mathbf{Q}^d$ we have $vF(ln^{-1})^{tr} = va^{-1}nFl^{tr}$ and hence $(L')^{\#} = L^{\#}an^{-1}$.

Since $L' \in \pi(L)$ one has $L' = L^{\#} \cap a^{-1}L$. Using this one obtains

$$Lan^{-2} = L'an^{-1} = L^{\#}an^{-1} \cap Ln^{-1} = (L')^{\#} \cap L' = L\,,$$

since $(L')^{\#}/L$ is the orthogonal complement of $L'/L$ in $L^{\#}/L$ with respect to the induced quadratic form with values in $\mathbf{Q}/\mathbf{Z}$. So $a^{-1}n^2 \in G$.

Finally we check that $n \in N$. Let $g \in G$, then $n^{-1}gn$ is in $G = \mathrm{Aut}(F)$ since $Ln^{-1}gn = L'gn = L'n = L$ and

$$n^{-1}gnFn^{tr}g^{tr}n^{-tr} = n^{-1}agFg^{tr}n^{-tr} = F\,. \qquad \square$$

## 4. Obtaining Elements of $N$

Now we give examples as to how one may construct elements $n$ of the normaliser $N$. To obtain similarities we are interested in $n \in N$ of determinant $\pm p^{d/2}$ for some (squarefree) natural number $p$ such that $p^{-1}n^2 \in G$. The first method is an application of the normaliser principle to the situation (iii) described in Section 2:

Proposition 4. *Let $U \trianglelefteq G$ be a normal subgroup of $G$ and assume that the commuting algebra $K := C_{M_d(\mathbf{Q})}(U)$ is isomorphic to a number field. If $c \in K$ satisfies $c^2 = p \in \mathbf{Q}^* I_d$, then $c$ lies in $N$.*

*Proof.* Since $G$ normalises $U$, it acts by conjugation (and hence as Galois automorphisms) on the abelian number field $K$. Now let $c \in K$, with $c^2 =: p \in \mathbf{Q}^* I_d$ and $g \in G$. Then $g$ stabilises the subfield $\mathbf{Q}[c]$ and hence $g^{-1}cg = \pm c$, which is equivalent to $c^{-1}gc = \pm g \in G$. Therefore $c \in N$, since we assumed that $-I_d \in G$. $\square$

The following construction described in [PlN 95] Proposition (II.4) also allows us to find elements of $N$.

For $i = 1, 2$ let $G_i \leq GL_{d_i}(\mathbf{Q})$ be finite rational irreducible matrix groups with commuting algebras $A_i \subseteq M_{d_i}(\mathbf{Q})$. Also let $Q$ be a maximal common subalgebra of dimension $z$ of $A_1$ and $A_2$. Let $d := \frac{d_1 d_2}{z}$ and view the $G_i$ as subgroups of $G_1 \underset{Q}{\otimes} G_2 \leq GL_d(\mathbf{Q})$. If there exist elements $a_i \in N_{GL_d(\mathbf{Q})}(G_i)$