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Artikel: THE LOCAL LINEARIZATION PROBLEM FOR SMOOTH SL(n) - ACTIONS

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REMARK 1.4. Notice that "linearizable" really means "locally linearizable". We don't consider the question of global linearizability since, even under the strongest hypotheses, global linearizability is too much to expect. For example, the action by conjugation of $PSL(2, \mathbf{R})$ on its universal cover $\widetilde{SL}(2, \mathbf{R}) \cong \mathbf{R}^3$ is analytic and locally linearizable, by the exponential map of the Lie algebra, but it is not globally linearizable because it has countably many fixed points (corresponding to the infinite discrete centre). In fact, even for algebraic actions, global linearization is not guaranteed [38]. Throughout this paper we will use the word *local* to mean "in some neighbourhood of the origin". We make the point however that in the case of a locally linearizable action, each homeomorphism of the action has its own domain on which it is linearizable, but there may be no common open domain for the entire group.

Note that we could also deal with *local group actions*; that is, maps Φ from some open neighbourhood of $(\mathrm{Id}, 0) \in G \times \mathbf{R}^m$ to some neighbourhood of $0 \in \mathbf{R}^m$ which satisfy the same conditions as for actions but only in the neighbourhood of $(\mathrm{Id}, 0) \in G \times \mathbf{R}^m$. There would be no essential changes in what follows.

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2. BACKGROUND AND MOTIVATION

The introduction to [21] begins: "The subject of smooth transformation groups has been strongly influenced by the following two problems: the smooth linearization problem (Is every smooth action of a compact Lie group on Euclidean space conjugate to a linear action ?), and the smooth fixed point problem (Does every smooth action of a compact Lie group on Euclidean space have a fixed point ?)." Indeed, for *compact* group actions, one has the following theorem of Salomon Bochner and Henri Cartan:

BOCHNER-CARTAN THEOREM (see [30, Chap. V, Theorem 1]). *For all $k = 1, \ldots, \infty$, every C^k-action of a compact group G on $(\mathbf{R}^m, 0)$ is C^k-linearizable.*

Proof. For each element $g \in G$, let $D(g)$ denote the differential of the action of g at the origin. Consider the map $F \colon \mathbf{R}^m \to \mathbf{R}^m$, defined by

$$F(x) = \int_G D(g)^{-1}\big(g(x)\big)\, d\mu$$

where μ is the normalized Haar measure on G. At the origin, the differential $D(F)$ is the identity map. So F is a local C^k-diffeomorphism in some neighbourhood of the origin. For each $h \in G$ one has

$$F\big(h(x)\big) = \int_G D(g)^{-1}\big(gh(x)\big)\, d\mu = \int_G D(gh^{-1})^{-1}\big(g(x)\big)\, d\mu$$
$$= \int_G D(h)\, D(g)^{-1}\big(g(x)\big)\, d\mu = D(h)\,\big(F(x)\big)\,,$$

for all $x \in \mathbf{R}^m$. So locally, F conjugates h to its linear part $D(h)$. □

REMARK 2.1. The same idea shows the following: suppose a group G acts on $(\mathbf{R}^m, 0)$ by C^k diffeomorphisms and contains a finite index subgroup G_0 which is C^k-linearizable. Then the action of G is C^k-linearizable. Indeed, we can assume that the action of G_0 is linear and we observe that $D(g)^{-1}\big(g(x)\big)$ depends only on the class $[g]$ of g in $G_0\backslash G$. Therefore we can define $F\colon \mathbf{R}^m \to \mathbf{R}^m$ by

$$F(x) = \sum_{[g]\in G_0\backslash G} D(g)^{-1}\big(g(x)\big)\,.$$

This F linearizes the action of G.

REMARK 2.2. The above theorem does not hold for C^0-actions. Indeed, here are two examples. First, recall that Bing constructed a continuous involution of S^3 whose fixed point set is the "horned sphere" [4] (see [5] for other examples). Removing one of these fixed points, one obtains a $\mathbf{Z}/2\mathbf{Z}$-action on $\mathbf{R}^3$ which is not locally topologically conjugate to a linear action, because the fixed point set is not locally flat.

Secondly, we give a non-linearizable action of $S^1 = SO(2)$, since we will be interested in $SO(n)$-actions later in the paper. Let M be any compact manifold with the same homotopy type as $\mathbf{C}P^n$, for some $n \geq 3$. By pulling back the Hopf fibration $S^1 \to S^{2n+1} \to \mathbf{C}P^n$, one obtains an S^1-bundle $\widehat{M} \to M$. Here $\widehat{M}$ is a compact manifold with the same homotopy type as S^{2n+1}, as one can see by applying the 5-lemma to the long exact homotopy sequence of the two fibrations. Hence, by Smale's proof of the generalized Poincaré conjecture [39], $\widehat{M}$ is homeomorphic to S^{2n+1}. Taking the cone of S^{2n+1} we obtain an S^1-action on $(\mathbf{R}^{2n+2}, 0)$. Locally, in a punctured neighbourhood of the origin, the orbit space of this action is homeomorphic to $M \times \mathbf{R}$. Now M may be chosen to be not homeomorphic to $\mathbf{C}P^n$ [15, 29]. Then, by the h-cobordism theorem [18, Essay 3], M is not h-cobordant to $\mathbf{C}P^n$ and consequently $M \times \mathbf{R}$ is not homeomorphic to $\mathbf{C}P^n \times \mathbf{R}$. Hence the S^1-action is not locally topologically conjugate to a linear action. Indeed, a linear action of $SO(2)$ on $\mathbf{R}^{2n+2}$ which is *free* outside the origin is linearly conjugate to a product of $n+1$ copies of the canonical action of $SO(2)$ on $\mathbf{R}^2$ and its local orbit space is homeomorphic to $\mathbf{C}P^n \times \mathbf{R}$.

In fact, for actions of noncompact groups, linearization results date back to Poincaré's work on analytic maps [34]. Recall that an element L of $GL(m, \mathbf{R})$ is called *hyperbolic* if all its eigenvalues $\lambda_1, \ldots, \lambda_m$ have modulus different from one. One says that L has a *resonance* if there is some relation of the form $\lambda_i = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_m^{k_m}$ where $1 \leq i \leq m$ and the k_j are non negative integers whose sum is bigger than 1. In the smooth case, one has the celebrated Sternberg Theorem:

THEOREM 2.3 ([43]). *In a neighbourhood of a fixed point, every C^∞-map whose linear part is hyperbolic and has no resonances is C^∞-linearizable.*

In the same vein, the Grobman-Hartman theorem says that in a neighbourhood of a hyperbolic fixed point, C^1-maps are topologically linearizable. See [17, Chap. 6] for a presentation of these results. Sternberg also proved in [43] that in a neighbourhood of a hyperbolic fixed point, every C^k-map whose linear part has no resonances is C^l-linearizable. Here l depends on the eigenvalues of the linear part and in general is less that k. According to [41], for the particular case of maps of the real line, one may take $l = k-1$. Here "hyperbolic" simply means that the derivative is a dilation (*i.e.* a linear map $x \mapsto ax$ with $|a| \neq 0, 1$). In fact, by [42, Theorem 4], for $k \geq 2$ one may take $l = k$ in this case. According to [12], even for $k = 1$, this last result is true but we could not locate a proof in the literature. All linearization results for maps pass immediately over to the case of flows, due the following lemma of Sternberg:

LEMMA 2.4 ([42, Lemma 4]). *Let $k = 1, \ldots, \infty$ and suppose that one has a C^k-flow ϕ^t on $(\mathbf{R}^m, 0)$. If ϕ^α is C^k-linearizable for some $\alpha \neq 0$, then ϕ^t is C^k-linearizable.*

Proof. Suppose that ϕ^α is linear. Then set

$$F(x) = \int_0^\alpha D(\phi^t)^{-1}\big(\phi^t(x)\big)\,dt\,,$$

and imitate the proof of the Bochner-Cartan Theorem. □

In particular, this gives the following result, which we will require later.

THEOREM 2.5. *Let $k = 1, \dots, \infty$ and suppose that one has a C^k-flow ϕ^t on $(\mathbf{R}, 0)$ whose linear part $D(\phi^1)$ is a dilation. Then ϕ^t is C^k-linearizable.*

According to Guillemin and Sternberg [11], it was Palais and Smale who suggested extending the Bochner-Cartan theorem to noncompact Lie groups. Indeed, analytic actions of semi-simple Lie groups are also linearizable, as proved by Kushnirenko [22], and independently by Guillemin and Sternberg [11] (see also [10, 20, 24, 26]). In particular, one has:

THEOREM 2.6. *Every analytic action of $SL(n,\mathbf{R})$ on $(\mathbf{R}^m, 0)$ is analytically linearizable.*

Proof. The proof that we sketch is slightly simpler that the one given in [11, 22]. It uses the famous unitary trick but does not use Poincaré's linearization theorem. First complexify the analytic $SL(n,\mathbf{R})$-action to obtain a local holomorphic action of $SL(n,\mathbf{C})$ on a neighbourhood of the origin in $\mathbf{C}^n$. Now restrict this action to the action of $SU(n)$. From the proof of the Bochner-Cartan theorem, we have on some neighbourhood U of the origin, a holomorphic map $F\colon (U,0) \to (\mathbf{C}^m, 0)$ such that

$$F\big(g(x)\big) = D(g)\,\big(F(x)\big), \quad \text{for all } g \in SU(n),\ x \in g^{-1}(U) \cap U\,,$$

where D is the differential of the action at the origin. Now fix $x \in \mathbf{C}^n$ and consider the set

$$S = \{g \in SL(n,\mathbf{C}) : F\big(g(x)\big) = D(g)\,\big(F(x)\big), \text{ on some neighbourhood of } 0\}\,.$$

This is a complex Lie subgroup of $SL(n,\mathbf{C})$ containing $SU(n)$. So, since $\mathfrak{sl}(n,\mathbf{C}) = \mathfrak{su}(n) \oplus i\,.\,\mathfrak{su}(n)$, one has $S = SL(n,\mathbf{C})$. Thus the action of $SL(n,\mathbf{C})$ is holomorphically linearizable.

Finally, F leaves $\mathbf{R}^m$ invariant and hence defines an analytic map which conjugates the action of $SL(n,\mathbf{R})$ to its linear part. □

Here is another important result:

THURSTON'S STABILITY THEOREM ([45]). *Let G be a connected Lie group or a finitely generated discrete group and suppose we have a non-trivial C^1-action of G on $(\mathbf{R}^m, 0)$. If G acts trivially on the tangent space $T_0\mathbf{R}^m$, then $H^1(G, \mathbf{R}) \neq 0$.*

REMARK 2.7. In the statement of Thurston's stability theorem, $H^*(G, \mathbf{R})$ denotes the continuous cohomology; so $H^1(G, \mathbf{R})$ is just the space of continuous homomorphisms from G to $\mathbf{R}$. Since $SL(n, \mathbf{R})$ is a simple Lie group, one has $H^1\big(SL(n, \mathbf{R}), \mathbf{R}\big) = 0$, for all n. For $n \geq 3$, $SL(n, \mathbf{Z})$ is a perfect group [32, Theorem VII.5], and so $H^1\big(SL(n, \mathbf{Z}), \mathbf{R}\big) = 0$. More generally, if Γ is a lattice in $SL(n, \mathbf{R})$, for some $n \geq 3$, then Γ has Kazhdan's property T and so $H^1(\Gamma, \mathbf{R}) = 0$ (see [50, Theorem 7.1.4 and Corollary 7.1.7]). If Γ is a lattice in $SL(2, \mathbf{R})$, then Γ doesn't have Kazhdan's property T (see [25, Proposition 3.1.9]) and $H^1(\Gamma, \mathbf{R})$ may be zero or non-zero, depending upon Γ. However, $H^1\big(SL(2, \mathbf{Z}), \mathbf{R}\big) = 0$, as the derived subgroup of $SL(2, \mathbf{Z})$ has finite index.

Note that the previous theorem can be regarded as a linearization result: for $G = SL(n, \mathbf{R})$, since $H^1(G, \mathbf{R}) = 0$, it says that the action is linearizable (trivially) if the differential at the origin is trivial.

The main point of this paper is to discuss to what extent the following theorem of Hermann can be generalized.

THEOREM 2.8 ([13]). *Every C^∞-action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^m, 0)$ is formally linearizable.*

Before proving this theorem, let us recall some concepts and notation. Firstly, if $i = (i_1, \dots, i_m)$, where $i_1, \dots, i_m \geq 0$, and if $x = (x_1, \dots, x_m) \in \mathbf{R}^m$, then we write $|i| = \sum_{j=1}^m i_j$ and we denote $\prod_{j=1}^m x_j^{i_j}$ by x^i. Now consider a formal power series

$$f(x) = \sum_i f_i x^i,$$

where $f_i \in \mathbf{R}^m$ for each i, and suppose that f has zero constant term (that is, $f(0) = 0$). The k^{th} Taylor polynomial of f is $T^k f = \sum_{|i| \leq k} f_i x^i$. We say that such a formal power series is a *formal diffeomorphism* of $(\mathbf{R}^m, 0)$ if $T^1 f$ defines a nonsingular linear map on $\mathbf{R}^m$. Let $\widehat{\text{Diff}}(\mathbf{R}^m, 0)$ denote the group of formal diffeomorphisms of $\mathbf{R}^m$. Note that Taylor expansion defines a natural homomorphism

$$\chi \colon \text{Diff}(\mathbf{R}^m, 0) \to \widehat{\text{Diff}}(\mathbf{R}^m, 0)$$

which is not injective, but is surjective [31, Chap. I, p. 28]. We say that a group $G \subset \operatorname{Diff}(\mathbf{R}^m, 0)$ is *formally linearizable* if there exists $f \in \widehat{\operatorname{Diff}}(\mathbf{R}^m, 0)$ such that f conjugates $\chi(G)$ to its linear part.

Proof of Theorem 2.8. Suppose we have a C^∞-action

$$\Phi\colon SL(n, \mathbf{R}) \to \operatorname{Diff}(\mathbf{R}^m, 0).$$

Let $\phi = \chi \circ \Phi$ and let $D\colon SL(n, \mathbf{R}) \to GL(m, \mathbf{R})$ be the linear part of ϕ; that is D is the homomorphism: $D(g) = T^1\phi(g)$. The proof is an inductive argument. First set $h_1 = \operatorname{Id} \in \widehat{\operatorname{Diff}}(\mathbf{R}^m, 0)$. Then for some integer $l > 1$ suppose that one has $h_{l-1} \in \widehat{\operatorname{Diff}}(\mathbf{R}^m, 0)$ such that for each $g \in SL(n, \mathbf{R})$, the Taylor polynomial $T^{l-1}\big(h_{l-1}\phi(g)h_{l-1}^{-1}\big)$ is linear and equals $D(g)$; that is, setting $g_{l-1} = h_{l-1}\phi(g)h_{l-1}^{-1}$, one has $T^{l-1}(g_{l-1}) = D(g)$. Let $E_l(g)$ denote the homogeneous part of g_l of degree l. Clearly E_l is a function on $SL(n, \mathbf{R})$ with values in the space P_l of homogeneous polynomials of degree l with values in $\mathbf{R}^m$. In terms of the group operation in $SL(n, \mathbf{R})$, we have

$$E_l(gh) = E_l(g) \circ D(h) + D(g) \circ E_l(h). \tag{1}$$

Notice that $SL(n, \mathbf{R})$ acts linearly on P_l; explicitly, for each $p \in P_l$ and each $g \in SL(n, \mathbf{R})$, one sets

$$g \cdot p = D(g) \circ p \circ D(g^{-1}).$$

So we can consider the cohomology of $SL(n, \mathbf{R})$ with values in P_l, twisted by this action. Now let $c_l(g) = E_l(g) \circ D(g^{-1})$ and observe that from (1), c_l is a 1-cocycle; that is:

$$c_l(gh) = g \cdot c_l(h) + c_l(g).$$

By Whitehead's lemma (see for example [14, Chapter VII.6]),

$$H^1\big(\mathfrak{sl}(n, \mathbf{R}), \mathbf{R}^m\big) = 0,$$

and hence by Van Est's Theorem [49], $H^1\big(SL(n, \mathbf{R}), \mathbf{R}^m\big) = 0$. So c_l is exact. Thus $c_l = dp_l$, for some $p_l \in P_l$; that is, $c_l(g) = g \cdot p_l - p_l$, for all $g \in SL(n, \mathbf{R})$. Consider the polynomial diffeomorphism $\eta = \operatorname{Id} + p_l \in \widehat{\operatorname{Diff}}(\mathbf{R}^m, 0)$. Note that

$$\eta^{-1} = \operatorname{Id} - p_l + \text{terms of order} > l.$$

Consider the conjugation of g_l by η. Modulo terms of order $> l$, one has:

$$\begin{aligned}
\eta g_l \eta^{-1} &\equiv (\operatorname{Id} + p_l) \circ \big(D(g) + E_l(g)\big) \circ (\operatorname{Id} - p_l) \\
&\equiv D(g) - D(g) \circ p_l + E_l(g) + p_l \circ D(g) \\
&\equiv D(g) - D(g) \circ p_l + c_l(g) \circ D(g) + p_l \circ D(g) \\
&\equiv D(g).
\end{aligned}$$

So, setting $h_l = \eta h_{l-1}$, we have that $T^l(h_l g h_l^{-1}) = D(g)$, for every $g \in SL(n, \mathbf{R})$. By induction, we have elements $h_l \in \widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ such that $T^l(h_l g h_l^{-1}) = D(g)$ for all $l > 0$. Finally set $h = \lim_{l \to \infty} h_l$. This makes sense in $\widehat{\mathrm{Diff}}(\mathbf{R}^m, 0)$ and by construction, h formally linearizes the action Φ.

3. Preparatory results

First let us make some general comments:

Remark 3.1. If a Lie group G acts on a topological manifold, then the restriction of the action to each orbit is a transitive G-action; that is, each orbit is a homogeneous space G/H for some closed subgroup $H \subset G$. In particular, transitive C^0-actions of $SL(n, \mathbf{R})$ are conjugate to analytic $SL(n, \mathbf{R})$-actions.

Remark 3.2. Every non-trivial continuous action of $SL(n, \mathbf{R})$ is either faithful, or factors through a faithful action of $PSL(n, \mathbf{R})$. Indeed, not only is $SL(n, \mathbf{R})$ simple as a Lie group (that is, its proper normal subgroups are discrete), but when n is odd it is simple as an abstract group and when n is even $PSL(n, \mathbf{R}) = SL(n, \mathbf{R})/\{\pm 1\}$ is simple as an abstract group. In particular, if n is odd, every non-trivial continuous action of $SL(n, \mathbf{R})$ is faithful. If n is even, non-faithful $SL(n, \mathbf{R})$-actions are common: see, for example, the adjoint action of $SL(n, \mathbf{R})$ for n even, or the irreducible $SL(2, \mathbf{R})$-representation on $\mathbf{R}^{2p+1}$ (see Section 5).

Remark 3.3. Every non-trivial C^1-action of $SL(n, \mathbf{R})$ on $(\mathbf{R}^n, 0)$ is faithful. Indeed, the differential at the origin defines a homomorphism $D \colon SL(n, \mathbf{R}) \to GL(n, \mathbf{R})$. In fact, since $SL(n, \mathbf{R})$ is a simple Lie group, the image of D is contained in $SL(n, \mathbf{R})$. By Thurston's stability theorem, D can't be trivial. So, for dimension reasons, D maps onto $SL(n, \mathbf{R})$. If an $SL(n, \mathbf{R})$-action is not faithful, then by the previous Remark, n is even and the element -1 acts trivially. But then D defines a homomorphism from $PSL(n, \mathbf{R})$ onto $SL(n, \mathbf{R})$, which is impossible since $PSL(n, \mathbf{R})$ is simple.