9. A C^∞ -ACTION OF SL(3,R) WHICH IS NOT LINEARIZABLE

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Notice that the vector fields $\bar{X}, \bar{Y}, R$ are linearly independent wherever $a \neq 0$. Indeed, putting $v = \sqrt{x^2 + y^2}$, one has:

$$\det(\bar{X}, \bar{Y}, R) = \det \begin{pmatrix} x^2 A(z, v) & z + xyA(z, v) & y + xzA(z, v) \\ z - xyA(z, v) & -y^2 A(z, v) & x - yzA(z, v) \\ -y & x & 0 \end{pmatrix}$$
$$= -(v^2 - z^2)\, v^2 A(z, v) = -(v^2 - z^2)\, a(v^2 - z^2).$$

It follows that if the function a is non-zero on $\mathbf{R}^+$, then the set of hyperbolic points in $\mathfrak{sl}(2, \mathbf{R})$ constitute a single orbit under the new action of $SL(2, \mathbf{R})$. Since no linear action of $SL(2, \mathbf{R})$ in $\mathbf{R}^3$ has an orbit of dimension 3, we conclude that our new action of $SL(2, \mathbf{R})$ is not linearizable. Note that outside the open orbit, this action coincides with the adjoint linear action.

In order to motivate the construction that we shall present in the next section, we now present another way of describing the non-linearizable action that we just constructed. Consider the subgroup Diag of $SL(2, \mathbf{R})$ of diagonal matrices and consider the trivial action of Diag on the positive line $\mathbf{R}^+_*$. It is easy to see that the suspension of this action is conjugate to the adjoint action of $SL(2, \mathbf{R})$ outside the invariant cone in $\mathbf{R}^3$. Now, since Diag is isomorphic to $\mathbf{R} \times \mathbf{Z}/2\mathbf{Z}$, it is easy to let Diag act non-trivially on $\mathbf{R}^+_*$ and the new suspension will provide a new action of $SL(2, \mathbf{R})$. If the new action of Diag extends to $\mathbf{R}^+$ and is sufficiently flat at 0, this action of $SL(2, \mathbf{R})$ can be equivariantly glued to the invariant cone and provides non-linearizable smooth actions of $SL(2, \mathbf{R})$ on $(\mathbf{R}^3, 0)$.

9. A C^∞-ACTION OF $SL(3, \mathbf{R})$ WHICH IS NOT LINEARIZABLE

We start with the adjoint action of $SL(3, \mathbf{R})$ on its Lie algebra $\mathfrak{sl}(3, \mathbf{R}) \cong \mathbf{R}^8$. Denote by Diag the subgroup of $SL(3, \mathbf{R})$ of diagonal matrices. This group is isomorphic to $\mathbf{R}^2 \times (\mathbf{Z}/2\mathbf{Z})^2$. Let *diag* $\subset \mathfrak{sl}(3, \mathbf{R})$ denote the 2-dimensional subalgebra consisting of diagonal matrices. The Weyl group, which is in this case the symmetric group on 3 letters, acts linearly on *diag* by permutation of the axis. The orbit of any point in *diag* under the adjoint action is a properly embedded submanifold of $\mathfrak{sl}(3, \mathbf{R})$ which intersects *diag* on some orbit of the Weyl group. Let C be a Weyl chamber in *diag*, for example the region consisting of diagonal matrices $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_1 \leq \lambda_2 \leq \lambda_3$. This is a fundamental domain for the action of the Weyl group.

Choose a closed disc D contained in the *interior* of the Weyl chamber C. The saturation $\mathrm{Sat}(D)$ of D under the adjoint action of $SL(3,\mathbf{R})$ is a properly embedded submanifold with boundary, which fibres over D.

LEMMA 9.1. *The action obtained by suspension of the trivial action of* Diag *on the disc D is the adjoint action of $SL(3,\mathbf{R})$ on the saturation of the disc.*

Proof. This is clear since the stabilizer of any point in D under the adjoint action is precisely Diag. □

LEMMA 9.2. *There exists a C^∞-action of $\mathbf{R}^2$ on the plane with support inside the unit disc.*

Proof. Let $\rho\colon \mathbf{R}^+ \to [0,1[$ be a C^∞-diffeomorphism which is equal to the identity in a neighbourhood of 0. This defines an embedding of $\mathbf{R}^2$ in the unit disc in $\mathbf{R}^2$ sending the point of polar coordinates (r,θ) to $\big(\rho(r),\theta\big)$. Now define an action of $\mathbf{R}^2$ on $\mathbf{R}^2$ in the following way. Inside the unit disc, this action is conjugated by the previous embedding to the canonical action of $\mathbf{R}^2$ on itself by translations. Outside, the action of $\mathbf{R}^2$ is trivial. It is a simple exercise to check that with a suitable choice of ρ, one can guarantee that this action is C^∞. □

Consider such a C^∞-action of $\mathbf{R}^2$ on D whose support lies in the interior of D. This defines an action of $\mathrm{Diag} \simeq \mathbf{R}^2 \times (\mathbf{Z}/2\mathbf{Z})^2$ on D for which $(\mathbf{Z}/2\mathbf{Z})^2$ acts trivially. By suspension, we get a C^∞-action of $SL(3,\mathbf{R})$ on some 8-manifold with boundary, which fibres over $SL(3,\mathbf{R})/\mathrm{Diag}$ with a closed disc as a fibre. This manifold is therefore diffeomorphic to the saturation $\mathrm{Sat}(D)$ of D under the adjoint action of $SL(3,\mathbf{R})$. The idea is to replace the adjoint action by this new action inside this manifold. However, since $\mathrm{Sat}(D)$ does not accumulate at the origin, this new action has not been modified near the origin and is therefore still locally linear. We shall therefore perform this modification on a sequence of discs in C accumulating to the origin and it will be easy to see that the action obtained in this way is not linearizable.

In order to realize this construction, we employ a family of C^∞-actions of $\mathbf{R}^2$ with support in the interior of the disc D, which depend continuously on a parameter $\epsilon \in [0,1]$ in the C^∞-topology and which is trivial when $\epsilon = 0$. This is easy to construct: just multiply the fundamental vector fields of some action of $\mathbf{R}^2$ by ϵ. We can also consider this family of actions as an action of $\mathbf{R}^2$ on $D \times [0,1]$ which is trivial on $D \times \{0\}$. By suspension, we get an action of $SL(3,\mathbf{R})$ on a 9-manifold, fibring over $SL(3,\mathbf{R})/\operatorname{Diag}$ with compact fibres $D \times [0,1]$. It is therefore diffeomorphic to $\operatorname{Sat}(D) \times [0,1]$. Therefore, we can project everything to $\operatorname{Sat}(D)$ in such a way that we get a continuous family of actions Φ_ϵ of $SL(3,\mathbf{R})$ on $\operatorname{Sat}(D)$ having the following properties. In some neighbourhood of the boundary of $\operatorname{Sat}(D)$, all these actions coincide with the adjoint action and Φ_0 is the adjoint action.

We can restate this as follows. Choose a basis of the Lie algebra $\mathfrak{sl}(3,\mathbf{R})$ and consider the associated linear vector fields $X_1, \ldots, X_8$ on $\mathfrak{sl}(3,\mathbf{R}) \cong \mathbf{R}^8$ which are the corresponding infinitesimal generators of the adjoint action. Then we have C^∞ families of vector fields $X_1^\epsilon, \ldots, X_8^\epsilon$ on $\operatorname{Sat}(D)$ such that $X_1^0 = X_1, \ldots, X_8^0 = X_8$ and such that they satisfy the same bracket relations for all ϵ; that is, they generate an action of $\mathfrak{sl}(3,\mathbf{R})$. Denote by R_i^ϵ the difference $X_i^\epsilon - X_i$ (for $i = 1, \ldots, 8$). Extending by 0 outside $\operatorname{Sat}(D)$, one gets C^∞ vector fields in $\mathbf{R}^8$.

Now consider some contracting homothety Λ of $\mathbf{R}^8$ which is such that all images of the disc D under the iterates of Λ are pairwise disjoint. Note that Λ preserves each X_i since these vector fields are linear. If a sequence $(\epsilon_l)_{l \geq 0} \in [0,1]$ converges to 0 sufficiently quickly when l goes to infinity then the infinite sums

$$\bar{R}_i = \sum_{l \geq 0} \Lambda_*^l (R_i^{\epsilon_l})$$

converge uniformly on compact sets in $\mathbf{R}^8$ and define C^∞ vector fields. Now, define the vector fields $\bar{X}_i = X_i + \bar{R}_i$. In $\Lambda^l\big(\operatorname{Sat}(D)\big)$ the vector fields $\bar{X}_i$ coincide with $\Lambda_*^l(X_i^{\epsilon_l})$ and outside these regions, they are equal to X_i. It follows that the $\bar{X}_i$ satisfy the same bracket relations as the X_i, and hence generate a C^∞-action of the Lie algebra $\mathfrak{sl}(3,\mathbf{R})$ on $\mathbf{R}^8$. These vector fields are complete and generate an action of $SL(3,\mathbf{R})$ since we know that inside $\Lambda^l\big(\operatorname{Sat}(D)\big)$ they integrate to a suspension and outside they integrate to the adjoint action.

The C^∞-action of $SL(3,\mathbf{R})$ on $\mathbf{R}^8$ thus obtained is not linearizable, since it has a countable number of open 8-dimensional orbits and this is obviously not possible for a linear representation.