7. Chen's proof of Vincent's theorem

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7. Chen's proof of Vincent's theorem

The proof given by Chen in 1987 [17] has the merit of focusing on the fractional linear transformations of the complex plane into itself as one of the principal tools involved in Vincent's theorem. Keeping the previous notation, we observe that the variable substitution

$$x \leftarrow \frac{a+bx}{1+x}$$

whose effect we have considered in detail, corresponds to the map $\mathcal{F}\colon \mathbf{C} \to \mathbf{C}$ defined by

$$y = \mathcal{F}(x) = \frac{x-a}{b-x}\,. \tag{7.1}$$

Chen's proof, which also carries over to the case of multiple roots, depends on a careful consideration of the effect of the map (7.1) on the roots of a polynomial.

Another essential tool is given by Obreschkoff's generalization of Descartes' rule of signs which may be stated as follows:

THEOREM 7.1 [30, p. 84]. *The number of roots of a real algebraic equation of degree n with V variations, whose argument φ verifies the inequality*

$$-\frac{\pi}{n-V} < \varphi < \frac{\pi}{n-V}\,,$$

is equal to V or is less than V by an even number.

We list some properties of the map (7.1) we are going to use.

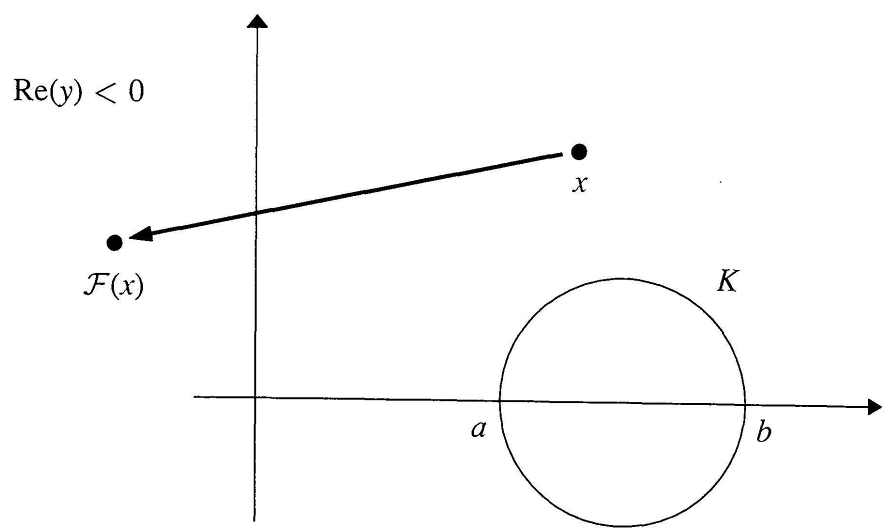

FIGURE 1a

- If $x \in \mathbf{R}$ then $\mathcal{F}(x) \in \mathbf{R}$ and, more precisely, if $x \in (a, b)$ then $\mathcal{F}(x) \in \mathbf{R}^+$.
- The map (7.1) transforms the circle K

$$\left|x - \frac{a+b}{2}\right| = \frac{1}{2}|b-a|$$

into the line $\operatorname{Re}(y) = 0$, and the exterior of this circle into the half-plane $\operatorname{Re}(y) < 0$ (see Fig. 1a).
- The map (7.1) (see Fig. 1b) transforms the left half plane $\operatorname{Re}(x) < 0$ into the interior of the circle H

$$\left|x + \frac{a+b}{2b}\right| = \frac{|a-b|}{2b}$$

whose diameter endpoints are -1 and $-\frac{a}{b}$. The imaginary axis is transformed into H.

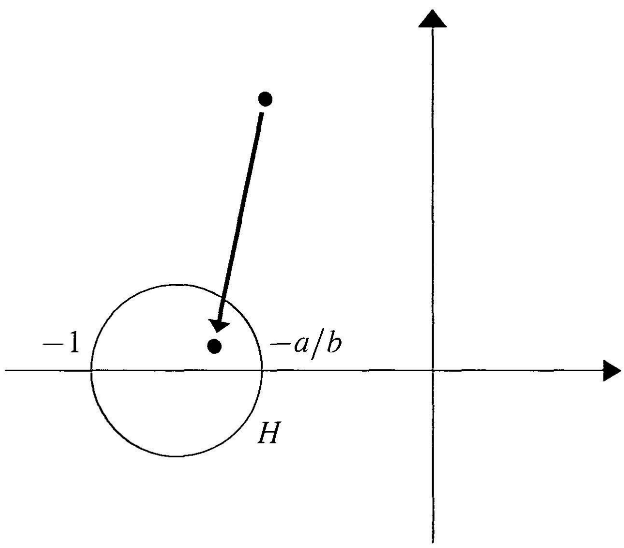

FIGURE 1b

With the help of these observations, we can prove the following

THEOREM 7.2 (Chen's Theorem 1). *Let $f(x)$ be a real polynomial of degree n whose least roots distance is Δ, and let $\gamma = [c_0, c_1, \ldots]$ with c_i non-negative integers, be a continued fraction, whose k-th convergent is denoted by $\frac{p_k}{q_k}$. Suppose that $F_h F_{h-1} \Delta > 1$. Then the polynomial*

$$f_{h+1}(x) = (q_{h-1} + q_h x)^n f\left(\frac{p_{h-1} + p_h x}{q_{h-1} + q_h x}\right)$$

has at most one root in the right half plane: one root if $f_{h+1}(x)$ has a positive number of variations, and no root if it has no variations.

Proof. We set, as before, $a = \dfrac{p_{h-1}}{q_{h-1}}$, $b = \dfrac{p_h}{q_h}$ and make the change of variable $x \leftarrow \dfrac{q_{h-1}}{q_h} x$. Once again we are led to study the number of variations of

$$\phi(x) = (1+x)^n f\left(\frac{a+bx}{1+x}\right).$$

Since

$$|b-a| = \frac{1}{q_h q_{h-1}} \le \frac{1}{F_h F_{h-1}} < \Delta$$

at most one root of $f(x)$ (which is necessarily real) may be in the interior of K.

If no root is in the interior of K, all the roots are mapped by $\mathcal{F}$ into the left half plane, and $\phi(x)$ has only factors with positive coefficients, and consequently has no variations. If a root x_0 (> 0) is in the interior of K, then $\mathcal{F}(x_0)$ is a positive real number and $\phi(x)$, having a positive root, must have a positive number of variations. $\square$

Chen is now in a position to prove the following theorem.

THEOREM 7.3 (Chen's Theorem 2). *Suppose that the real polynomial $f(x)$ of degree n has only one root x_0 in the right half plane, and consider the continued fraction $\gamma = [c_0, c_1, \dots]$ with c_i non-negative integers. Suppose that the integer h is sufficiently large to have*

$$\min(p_h q_{h-1}, p_{h-1} q_h) > \frac{n}{6}.$$

If the polynomial $f_{h+1}(x)$ has V variations, then V is exactly the multiplicity of x_0 and $x_0 \in \left(\dfrac{p_{h-1}}{q_{h-1}}, \dfrac{p_h}{q_h}\right)$.

Proof. To avoid trivial cases, we suppose that $n \ge 3$ and that $V \ge 3$. We substitute $\phi(x)$ for $f_{h+1}(x)$ as in the previous theorem.

By hypothesis f has only one (real) root x_0 in the right half plane. If $x_0 \notin \left(\dfrac{p_{h-1}}{q_{h-1}}, \dfrac{p_h}{q_h}\right)$ then $\phi(x)$ has no variations: hence a contradiction. Therefore, $x_0 \in \left(\dfrac{p_{h-1}}{q_{h-1}}, \dfrac{p_h}{q_h}\right)$ and the multiplicity of the root $\mathcal{F}(x_0)$ of $\phi(x)$ is smaller than V. Since $\mathcal{F}$ is one-to-one it follows that the multiplicity of x_0 is also smaller than V.

By the fundamental theorem of algebra the number of roots of $f(x)$ in the left half plane or on the imaginary axis is greater than $n - V$.

$\mathcal{F}$ transforms all the roots different from x_0 into or on the circle H, hence $\phi(x)$ has a positive real root $\mathcal{F}(x_0)$ and all the other roots, whose number is greater than $n - V$, are inside H or on its circumference.

Let $g(x) = \phi(-x)$ and denote by $-H$ the circle symmetric to H with respect to the imaginary axis. The polynomial $g(x)$ has at least $n - V$ roots inside $-H$ or on its circumference. Denoting by V' the number of variations of $g(x)$ we have

$$V' \leq n - V .$$

We prove that the number of the roots of $g(x)$ within $-H$ (or on its boundary) is exactly $n - V$.

From

$$\min(p_{h-1}\, q_h, p_h\, q_{h-1}) > \frac{n}{6} ,$$

we have

$$\frac{1}{2\, p_{h-1}\, q_h} < \frac{3}{n} \quad \text{and} \quad \frac{1}{2\, p_h\, q_{h-1}} < \frac{3}{n} .$$

Hence

$$\frac{3}{n} < \frac{\pi}{n} < \tan\frac{\pi}{n} .$$

It follows that

$$\frac{1}{2\, p_{h-1}\, q_h} < \tan\frac{\pi}{n} , \quad \frac{1}{2\, p_h\, q_{h-1}} < \tan\frac{\pi}{n} .$$

The maximum absolute value of the tangent of the argument of a point inside the circle $-H$ is given by

$$\frac{|a-b|}{2\sqrt{ab}} = \frac{1}{2\sqrt{p_h\, q_{h-1}\, p_{h-1}\, q_h}} ,$$

and

$$\frac{1}{2\sqrt{p_h\, q_{h-1}}}\, \frac{1}{\sqrt{p_{h-1}\, q_h}} \leq \frac{1}{2 \min(p_h\, q_{h-1}, p_{h-1}\, q_h)} .$$

It follows that the circle $-H$ is contained in the sector

$$W = \left\{ x \,:\, |\arg(x)| < \frac{\pi}{n} \right\} .$$

The polynomial $g(x)$ has degree n and it has V' variations. Since $V' < n$, we have

$$\frac{\pi}{n - V'} > \frac{\pi}{n} .$$

We may apply Obreschkoff's result to conclude that the number of roots of $g(x)$ within the sector

$$W' = \left\{x : |\arg(x)| < \frac{\pi}{n - V'}\right\}$$

is less than $V' \leq n - V$. But since $g(x)$ has at least $n - V$ roots within or on the boundary of $-H \subseteq W \subseteq W'$, the number of roots of $g(x)$ within or on $-H$ is exactly $n - V$. Then $\phi(x)$ and therefore $f_{h+1}(x)$ has exactly $n - V$ roots in the left half plane. Hence V is the multiplicity of the only positive root of $f_{h+1}(x)$ and therefore of $f(x)$. □

THEOREM 7.4. *Let $f(x)$ be an integral polynomial of degree $n \geq 3$, with only one root x_0 in the right half plane, and suppose it has at least 3 variations. Let m be the smallest integer such that*

$$m > \tfrac{1}{2}\log_\phi n,$$

where $\phi = \frac{1+\sqrt{5}}{2}$. Let $\gamma = [c_0, c_1, \ldots]$ with c_i positive integers. If V is the number of variations of f_{m+1}, then the root x_0 has multiplicity V and $x_0 \in \left(\frac{p_{m-1}}{q_{m-1}}, \frac{p_m}{q_m}\right)$.

Proof. Since

$$m > \tfrac{1}{2}\log_\phi n,$$

we have

$$\phi^{2m} > n.$$

Let $\psi = \frac{1-\sqrt{5}}{2}$. Writing the n-th Fibonacci number F_n in our notation (see note 4) as

$$F_n = \frac{1}{\sqrt{5}}(\phi^{n+1} - \psi^{n+1}),$$

we easily deduce

$$F_{m-1}^2 \geq \frac{n}{6}.$$

The hypothesis $c_i \geq 1$ implies that $p_k > q_k$ for every k, hence

$$\min(p_m q_{m-1}, p_{m-1} q_m) > F_{m-1}^2 \geq \frac{n}{6}$$

and we may apply the previous theorem[19]). □

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[19]) The reason why Chen does not explicitly require $F_{m-1}^2 > n/6$ is not clear, but we have followed his approach.

THEOREM 7.5 (Chen's Main Theorem). *Let $f(x)$ be an integral polynomial of degree $n \geq 3$ with at least 3 variations. Let h be the smallest positive integer for which*

$$F_{h-1}^2 \Delta > 1 ,$$

and let m be the smallest positive integer such that

$$m > \tfrac{1}{2} \log_\phi n .$$

Let $k = h + m$. For an arbitrary continued fraction $\gamma = [c_0, c_1, \dots] > 0$, consider the polynomial f_{k+1} constructed by F_k. If V is the number of variations of f_{k+1} then the polynomial f has a unique positive root in $\left(\frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k}\right)$ and V is its multiplicity.

Proof. After h steps, the polynomial f_{h+1} might have no variations, and then f_{k+1} will have no variations. If f_{h+1} has V variations, by Chen's Theorem 1 it has a positive root in the right half plane. The partial quotients c_i are ≥ 1 for $i > h$, and so we may apply Chen's Theorem 2. □

8. A NEW PROOF OF VINCENT'S THEOREM

In this section we give a new and simpler proof of Vincent's theorem, which in turn improves on Chen's result. For the sake of clarity, we prefer to deal separately with the two cases of simple and multiple roots.

8.1 THE CASE OF SIMPLE ROOTS

In the case of simple roots, we show that Vincent's theorem holds under the only assumption

$$\Delta F_h F_{h-1} > \frac{2}{\sqrt{3}}$$

independently of the polynomial degree n.

Our proof depends on the following result by Obreschkoff [30, p. 81].

LEMMA 8.1. *Let $f(x)$ be a real polynomial with V variations in the sequence of its coefficients; let V_1 be the number of variations of the polynomial $f_1(x) = (x^2 + 2\rho x \cos\varphi + \rho^2) f(x)$ (where $\rho > 0$ and $|\varphi| < \frac{\pi}{V+2}$).*
Then $V \geq V_1$, and the difference $V - V_1$ is an even number.