§2. Local properties of $(G)_n$

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **44 (1998)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

Then we define the *rotation number* $\mathrm{rot}(\sigma)$ to be

$$\sum_C \sigma(C)\,\mathrm{rot}(C)$$

where the sum is over all simple closed curves C equipped with $\sigma(C) \in \mathcal{N}$ and $\mathrm{rot}(C)$ is the rotation number of C (i.e., 1 if C is counter-clockwise and -1 otherwise). For example $\mathrm{rot}(\sigma) = 2$ for the state described in Figure 1.2 (see Figure 1.4).

Now we define $\langle G \rangle_n$ as follows.

$$\langle G \rangle_n = \sum_{\sigma:\text{state}} \Big\{ \prod_{v:\text{vertex}} \mathrm{wt}(v;\sigma) \Big\} q^{\mathrm{rot}(\sigma)} .$$

We define $\langle \text{empty graph} \rangle_n = 1$. It is clear that this is invariant under ambient isotopy of $\mathbf{R}^2$. Note that our invariant can be regarded as a colored graph invariant introduced by N. Yu. Reshetikhin and V. G. Turaev in [14] replacing each vertex by a "coupon". The coupon with two legs in would correspond to a projection $V_i \otimes V_j \to V_{i+j}$ and that with two legs out to an inclusion $V_{i+j} \to V_i \otimes V_j$, where V_i is the irreducible representation of $SU(n)$ corresponding to the i-fold anti-symmetric tensor of the vector representation.

§2. Local properties of $\langle G \rangle_n$

We will describe some local properties of $\langle G \rangle_n$. In what follows diagrams indicated in each equality are identical outside the angle brackets $\langle\ \ \rangle_n$ and each equality also holds if we reverse all the orientations of diagrams in both hand sides. A number near an edge indicates its flow. If a flow in a diagram exceeds n, we disregard the term where the diagram appears.

We put

$$[k] = \frac{q^{k/2} - q^{-k/2}}{q^{1/2} - q^{-1/2}} ,$$

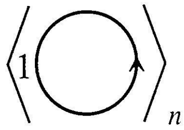
$$[k]! = [1][2] \cdots [k] ,$$

and

$$\begin{bmatrix} i \\ j \end{bmatrix} = \frac{[i]!}{[j]![i-j]!} .$$

In the following equations we mean that if we replace the graph appearing in the left hand side with the one in the right hand side, we obtain the same value.

LEMMA 2.1.



$= [n] \langle \varnothing \rangle_n .$

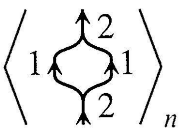
Proof. From the definition the left hand side is equal to

$$\sum_{i=-(n-1)/2}^{(n-1)/2} q^i ,$$

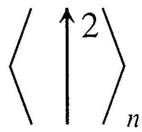
which is $[n]$, completing the proof. □

LEMMA 2.2.

(2.1)



$= [2]$



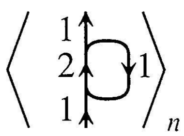
.

Proof. Consider a state σ of the left hand side. First note that the top-most and the bottom-most edges are equipped with the same subset. So we may put it $\{\alpha, \beta\}$ $(\alpha < \beta)$. Then there are two cases; (i) the left edge is equipped with $\{\alpha\}$ and the right one with $\{\beta\}$ and (ii) the left edge is equipped with $\{\beta\}$ and the right one with $\{\alpha\}$. In the first case the weights of the upper and lower vertex are the same and equal to $q^{1/4}$. In the second case they are also the same and equal to $q^{-1/4}$. Therefore the contribution of the two vertices is

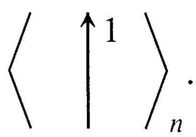
$$q^{1/2} + q^{-1/2} = [2] ,$$

which is independent of σ and the conclusion follows. □

LEMMA 2.3.



$= [n-1]$



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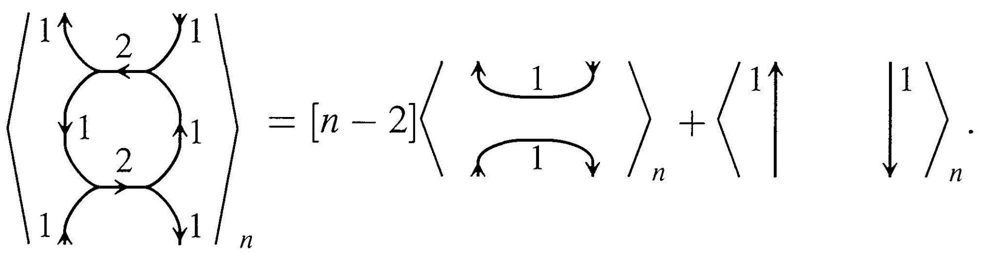
Proof. In this case the top-most and the bottom-most edges of the left hand side are equipped with the same subset for any state σ as the previous lemma. So we put it $\{\alpha\}$. If the right edge is equipped with $\{\beta\}$ $(\beta \neq \alpha)$, then the weights of the two vertices are the same and equal to $q^{\operatorname{sign}(\beta-\alpha)/4}$.

Since its contribution to the rotation number is $-\beta$, the contribution of the left hand side is

$$\sum_{\beta\neq\alpha} q^{\operatorname{sign}(\beta-\alpha)/2-\beta} = \sum_{\beta=-(n-1)/2}^{\alpha-1} q^{-1/2-\beta} + \sum_{\beta=\alpha+1}^{(n-1)/2} q^{1/2-\beta}$$
$$= q^{(n-2)/2} + q^{(n-4)/2} + \cdots + q^{-\alpha+1/2}$$
$$+ q^{-\alpha-1/2} + \cdots + q^{-(n-2)/2} = [n-1]$$

and the proof is complete. $\square$

LEMMA 2.4.



Proof. There are three possibilities:

(1) the two edges at the top are equipped with $\{\alpha\}$ and the two edges at the bottom are equipped with $\{\beta\}$ $(\alpha \neq \beta)$;

(2) both the top left and the bottom left edges are equipped with $\{\alpha\}$ and both the top right and the bottom right edges are equipped with $\{\beta\}$ $(\alpha \neq \beta)$;

and

(3) all the four edges at the corners are equipped with $\{\alpha\}$.

In the first case, the two horizontal edges are equipped with $\{\gamma\}$ $(\alpha \neq \gamma \neq \beta)$. So the contribution of the left hand side is

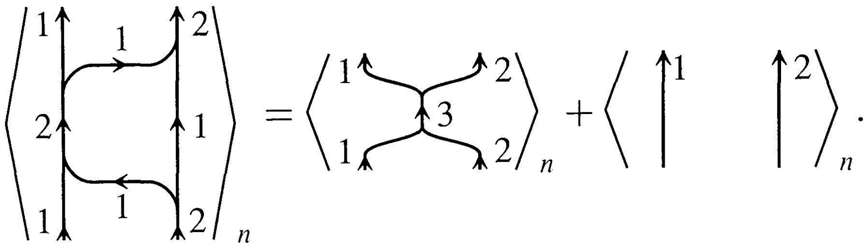
$$\Bigl(\sum_{\substack{\gamma \\ \alpha\neq\gamma\neq\beta}} q^{\operatorname{sign}(\alpha-\gamma)/2+\operatorname{sign}(\beta-\gamma)/2} q^{\gamma}\Bigr) q^{-(\alpha+\beta)/2} = [n-2]\, q^{-(\alpha+\beta)/2}$$

which is equal to the contribution of the right hand side.

In the second case, the contribution of the first term of the right hand side is zero. It is easy to see that the left hand side and the second term of the right hand side coincide.

In the third case, the contribution of the left hand side is $[n-2]\, q^{-\alpha} + 1$, which is equal to that of the right hand side. This completes the proof. $\square$

LEMMA 2.5.



Proof. There are three possibilities:

(1) both the top left and the bottom left edges are equipped with $\{\alpha\}$ and both the top right and the bottom right edges are equipped with $\{\beta, \gamma\}$ (α, β and γ are all distinct);

(2) the top left and the bottom left edges are equipped with $\{\alpha\}$ and $\{\beta\}$ respectively and the top right and the bottom right edges are equipped with $\{\beta, \gamma\}$ and $\{\alpha, \gamma\}$ respectively (α, β and γ are all distinct);

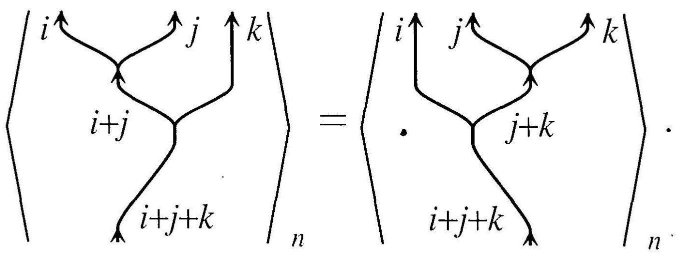
and

(3) both the top left and the bottom left edges are equipped with $\{\alpha\}$ and both the top right and the bottom right edges are equipped with $\{\alpha, \beta\}$ ($\alpha \neq \beta$).

In the first case both the upper and the lower horizontal edges in the left hand side are equipped with $\{\beta\}$ or both of them are equipped with $\{\gamma\}$. So the contribution of the left hand side is $q^{\mathrm{sign}(\beta-\alpha)/2+\mathrm{sign}(\gamma-\beta)/2} + q^{\mathrm{sign}(\gamma-\alpha)/2+\mathrm{sign}(\beta-\gamma)/2}$. On the other hand that of the right hand side is $q^{1-\pi(\{\alpha\},\{\beta,\gamma\})} + 1$. It can be easily checked that these are the same.

The second and the third cases are easily checked and left to the reader. □

LEMMA 2.6.



Proof. This follows from the fact that

$$
\begin{aligned}
&\#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\} + \#\{(a, a_3) \in (A_1 \cup A_2) \times A_3 \mid a > a_3\} \\
&\quad = \#\{(a_1, a_2) \in A_1 \times A_2 \mid a_1 > a_2\} \\
&\qquad + \#\{(a_1, a_3) \in A_1 \times A_3 \mid a_1 > a_3\} + \#\{(a_2, a_3) \in A_2 \times A_3 \mid a_2 > a_3\} \\
&\quad = \#\{(a_1, a) \in A_1 \times (A_2 \cup A_3) \mid a_1 > a\} + \#\{(a_2, a_3) \in A_2 \times A_3 \mid a_2 > a_3\}.
\end{aligned}
$$

□