# 7.1 A COMBINATORIAL PROOF

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$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d$, for $k_i \leq n_i - 1$ and $1 \leq i \leq d$ is conjectured to be of the form $c_d + \prod_{i=1}^{d-1} n_i$, where $c_d$ is a constant independent of $n_1, \ldots, n_d$.

## 7. FREQUENCIES OF FACTORS FOR BINARY CODINGS OF ROTATIONS

We will prove in this section the following result, which corresponds to the case $\min\{n_1, n_2\} = 2$ in Theorem 14. The idea of using a reflection of the unit circle can also be found in the original proof in [29].

THEOREM 18. *Let $\alpha$ be an irrational number in $]0,1[$, $\beta \neq 0$ a real number and $n$ a non-zero integer. The set of points $\{0\}, \{\beta\}, \{\alpha\}$, $\{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$ divides the circle into a finite number of intervals, whose lengths take at most five values.*

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We will prove Theorem 18 by introducing a coding of the rotation by angle $\alpha$ with respect to the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$.

Let $\alpha$ be an irrational number, $\beta$ a non-zero real number and $n$ an integer. Let $I_1, \ldots, I_p$ denote the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \ldots, \{n\alpha\}, \{\beta+n\alpha\}$. Let $u = (u_n)_{n \in \mathbf{N}}$ be the sequence defined on the alphabet $\Sigma = \{a_1, \ldots, a_p\}$ as the coding of the orbit of 0 under the rotation $R$ of angle $\alpha$ under the partition $\{I_1, \ldots, I_p\}$:

$$u_n = a_k \iff \{n\alpha\} \in I_k.$$

The frequency of the letter $a_k$ in the sequence $u$ is equal to the length of the interval $I_k$, by uniform distribution of the sequence $(\{n\alpha\})_{n \in \mathbf{N}}$. We must now prove that the frequencies of the letters of $u$ take at most five values. Let us consider the graph $\Gamma_1$ of words of $u$ of length 1. There is one edge from $a_k$ to $a_{k'}$ if $I_{k'}$ is the image of $I_k$ by the rotation $R$ or if $I_{k'}$ contains $\{-\alpha\}$ or $\{-\alpha + \beta\}$. Therefore the graph $\Gamma_1$ contains $p$ vertices (one for each letter) and $p + 2$ edges: indeed, every vertex has only one leaving edge, except the ones associated with the intervals containing $\{-\alpha\}$ or $\{n - \alpha + \beta\}$, which have two leaving edges (if both of these points belong to the same interval $I_k$, then $a_k$ has three leaving edges and all the other intervals have only one edge). In other words, we have $p(1) = p$ and $p(2) = p + 2$. As in the proof of Theorem 6, this implies that there are at most 6 branches in $\Gamma_1$: indeed, each

branch starts with a vertex with more than one entering edge (this provides at most two branches) or just after a vertex with at least two leaving edges (at most four branches are of this kind). We deduce from this that the frequencies of the letters in $u$ take at most 6 values. Let us prove that at least two branches of $\Gamma_1$ have the same frequency, which will complete the proof.

Let $s$ denote the reflection of the circle defined by $s: x \rightarrow \{\beta + n\alpha - x\}$. This reflection leaves invariant the endpoints of the intervals $I_1, \ldots, I_p$ and thus induces a permutation $\sigma$ of the interiors of the intervals $I_k$, which can also be seen as a permutation of $\Sigma$. The length of $I_k$ is equal to the length of $I_{\sigma(k)} = s(I_k)$. The frequency of the letter $a_k$ is thus equal to the frequency of the letter $\sigma(a_k)$. Note that if $a_i a_j$ is a factor of $u$, then $\sigma(a_j)\sigma(a_i)$ is also a factor. We deduce from this that if there is an edge in $\Gamma_1$ from $a_i$ to $a_j$, then there is also an edge from $\sigma(a_j)$ to $\sigma(a_i)$, or in other words, that $\Gamma_1$ is invariant by the following action of $\sigma$ : the image of the vertex associated with the letter $a$ is equal to the vertex associated with $\sigma(a)$ and the image of the edge $a \rightarrow b$ is the edge $\sigma(b) \rightarrow \sigma(a)$, i.e., each letter is replaced by its image and the direction of every edge is changed. Furthermore, the image of a branch is a branch. Let us prove that at most four branches of the graph $\Gamma_1$ are invariant by $\sigma$. Let $B = U_1 \rightarrow U_2 \rightarrow \cdots \rightarrow U_q$ be an invariant branch of the graph. We have $B = \sigma(B) = \sigma(U_q) \rightarrow \cdots \rightarrow \sigma(U_1)$. We thus get $\sigma(U_k) = U_{q+1-k}$.

- Suppose that there exists $i$ such that $U_i = \sigma(U_i)$. Therefore the interval $I_i$ must contain a fixed point for $s$. Since there are only two such fixed points, at most two branches can satisfy this property.

- Let us suppose that $U_i \neq \sigma(U_i)$ for each $1 \leq i \leq q$. We thus have $q$ even and $\sigma(U_{q/2}) = U_{q/2+1}$. Let $I$ (respectively, $I'$) be the closure of the interval associated with $U_{q/2}$ (respectively, $U_{q/2+1}$). We thus get $s(I) = I'$. Furthermore, $I'$ is the image of $I$ by the rotation $R$, because of the edge $U_{q/2} \rightarrow U_{q/2} + 1$. This implies that $I'$ contains a fixed point of the symmetry $s \circ R^{-1}$, which has at most two fixed points. Hence, at most two branches are of this kind.

We have proved that at most four basic paths can be their own image by $\sigma$. Therefore, there exist among the six branches at least two different branches, say $A$ and $B$, such that $B = \sigma(A)$. Thus $A$ and $B$ have the same frequency, which implies that there are at most five possible frequencies for the letters of $u$.