

# 8. The 3d distance theorem

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## 7.2 APPLICATION TO BINARY CODINGS

A more natural coding of the rotation  $R$  would have been with respect to the partition  $[0, \beta[, [\beta, 1[$ . The points  $\{0\}, \{\beta\}, \{\alpha\}, \{\beta + \alpha\}, \dots, \{n\alpha\}, \{\beta + n\alpha\}$  are the endpoints of the sets  $I(w_1, \dots, w_n)$ , following the notation of Section 2. But these sets might not be connected. Thus the frequencies of factors of length  $n$  are the sums of the lengths of the connected components of the sets  $I(w_1, \dots, w_n)$ . Despite this disadvantage, this coding allows us to deduce the following result from Lemma 3.

**THEOREM 19.** *Let  $u$  be a coding of an irrational rotation with respect to the partition into two intervals  $\{[0, \beta[, [\beta, 1[\}$ , where  $0 < \beta < 1$ . Let  $n^{(1)}$  denote the connectedness index of  $u$ . The frequencies of factors of given length  $n \geq n^{(1)}$  of  $u$  take at most 5 values. Furthermore, the set of factors of  $u$  is stable by mirror image, i.e., if the word  $a_1 \dots a_n$  is a factor of the sequence  $u$ , then  $a_n \dots a_1$  is also a factor and furthermore, both words have the same frequency.*

*Proof.* It remains to prove the part of this theorem concerning the stability by mirror image. Assume we are given a fixed positive integer  $n$ . Let  $s_n$  be the reflection of the circle defined by  $s_n: x \rightarrow \{\beta - (n-1)\alpha - x\}$ . We have  $s_n(R^{-k}(I_j)) = R^{(-n+1+k)}(I_j)$ , for  $j = 0, 1$ , following the previous notation. The image of  $I(w_1, \dots, w_n)$  by  $s_n$  is  $I(w_n, \dots, w_1)$ ; they thus have the same length, which gives the result.

**REMARK.** A study of the topology of the graph of words for a binary coding of an irrational rotation of complexity satisfying ultimately  $p(n+1) - p(n) = 2$  can be found in [24] or in [46].

## 8. THE $3d$ DISTANCE THEOREM

Following the idea of the above proof, let us give a combinatorial proof of the  $3d$  distance theorem.

**THE  $3d$  DISTANCE THEOREM.** *Assume we are given  $0 < \alpha < 1$  irrational,  $\gamma_1, \dots, \gamma_d$  real numbers and  $n_1, \dots, n_d$  positive integers. The points  $\{n\alpha + \gamma_i\}$ , for  $0 \leq n < n_i$  and  $1 \leq i \leq d$ , partition the unit circle into at most  $n_1 + \dots + n_d$  intervals, having at most  $3d$  different lengths.*

*Proof.* Let us consider a coding of the rotation by angle  $\alpha$  under the left-closed and right-open partition of the unit circle bounded by all the points of the form  $\{n\alpha + \gamma_i\}$ , for  $0 \leq n < n_i$  and  $1 \leq i \leq d$ ; let  $\beta_0, \dots, \beta_{p-1}$  denote these consecutive points. The letter associated with the interval  $I_k = [\beta_k, \beta_{k+1}[$  has a unique right extension, except when  $I_k$  contains points of the form  $\{\beta_i - \alpha\}$ . Suppose there are  $q \geq 2$  points of this form; the associated letter has  $q + 1$  right extensions. Since there are at most  $d$  points of this type, we obtain  $p(2) - p(1) \leq d$ . We deduce from Theorem 6 that there are at most  $3d$  different frequencies for the letters of the coding, i.e., there are at most  $3d$  different lengths for the intervals  $I_k$ .

REMARK. The start and finish intervals as introduced by Liang in his proof in [37] correspond exactly to the beginning of the branches in the graph of words. Indeed, Liang shows that any interval is associated either with a start point  $\{\gamma_i\}$  (i.e., with one extension of a factor having more than one right extension) or with a finish point  $\{(n_i - 1)\alpha + \gamma_i\}$  (i.e., with a factor having more than one left extension). Counting the finish and start points defined in [37] (there are  $3d$  such points) is equivalent to counting the number of branches in the graph of words.

As in the remark of the previous section, we can consider a coding of the rotation by irrational angle  $1 - \alpha$  under the partition  $\{[\gamma_1, \gamma_2[, \dots, [\gamma_d, \gamma_1[\}$ . For such a coding, the  $3d$  distance theorem can be rephrased as follows.

**THEOREM 20.** *The frequencies of the factors of given length  $n \geq n^{(1)}$  of a coding of a rotation by irrational angle under a partition in  $d$  intervals take at most  $3d$  values, where  $n^{(1)}$  denotes the connectedness index.*

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