4. Real structures

4. Real structures

Recall that a *real algebraic variety* is a pair (X, S) where X is a complex algebraic variety and $S\colon X \to X$ is an anti-holomorphic involution on it. The set of fixed points of S is the *real part* of (X, S). S acts on the group of divisors $\mathrm{Div}(X)$: if $D \in \mathrm{Div}(X)$ is defined locally by analytic functions f_α, then $S(D)$ is defined by the analytic functions $\overline{f_\alpha \circ S}$. Thus it is natural to define an involution S^* on the sheaf of analytic functions $\mathcal{O}_X$

$$S^*\colon \Gamma\big(S(U), \mathcal{O}_X\big) \to \Gamma\big(U, \mathcal{O}_X\big) \ : f \mapsto \overline{f \circ S}\,.$$

This also induces an involution on the groups of one-forms and one-cycles. If $\omega \in H^0(X, \Omega^1)$, $c \in H_1(X, \mathbf{Z})$, then $\int_c S^*\omega = \overline{\int_{S(c)} \omega}$. A form ω is S-real if and only if $S^*\omega = \omega$ and one may always choose a basis of S-real forms. In the case when $X = C_h$ is the spectral curve of the Lagrange top, the action of S on $\mathrm{Div}(X)$ induces an involution on $J(C_h; \infty^\pm)$. This, however, does not suffice to determine the real structure of the invariant manifold $T_h \sim J(C_h; \infty^\pm) \setminus \phi^{-1}(p)$ (Theorem 2.2), as it will also depend on the point $p \in J(C_h)$. Recall that the symmetric product $S^2\breve{C}_h$ is bi-rational to T_h. Thus the generalized Jacobian and the invariant manifold T_h are identified by the Abel map

$$\mathcal{A}\colon S^2\breve{C}_h \to J(C_h; \infty^\pm) \ : \ P_1 + P_2 \mapsto \int_{W_1+W_2}^{P_1+P_2} \omega \ , \qquad \omega = (\omega_1, \omega_2)\,. \tag{59}$$

This induces an involution on $J(C_h; \infty^\pm)$, $z \to S(z)$, where

$$z = \int_{W_1+W_2}^{P_1+P_2} \omega \ , \qquad S(z) = \int_{W_1+W_2}^{S(P_1+P_2)} \omega\,.$$

Of course this depends on the fixed points $W_1, W_2 \in J(C_h; \infty^\pm)$. Let ω_1, ω_2 be S-real . Then

$$S(z) = \int_{W_1+W_2}^{S(W_1+W_2)} \omega + \int_{S(W_1+W_2)}^{S(P_1+P_2)} \omega = \int_{W_1+W_2}^{S(W_1+W_2)} \omega + \overline{\int_{W_1+W_2}^{P_1+P_2} \omega} = S(0) + \bar{z}\,.$$

If S has a fixed point on $J(C_h; \infty^\pm)$ (this does not depend on W_1, W_2) then one may always choose it for origin, and hence $S(z) = \bar{z}$ becomes a group homomorphism.

Denote by S the anti-holomorphic involution on the spectral curve C_h defined by $S(\lambda, \mu) = (\bar{\lambda}, -\bar{\mu})$. This involution comes from the real Lax pair of Adler and van Moerbeke defined in Section 2. We shall also suppose that the real polynomial $f(\lambda)$ has distinct roots. S induces an involution on the usual Jacobian $J(C_h)$ which we also denote by S, and an involution on the generalized Jacobian $J(C_h;\infty^{\pm})$ which we denote by S^+. If we use (59), then in terms of the Jacobi polynomials U, V, W, it is given by

$$S^+ : (U, V, W) \mapsto (\overline{U}, -\overline{V}, \overline{W}).$$

There is another natural anti-holomorphic involution on T_h given by the usual complex conjugation

$$\left(\Omega_i, \Gamma_i\right) \mapsto \left(\overline{\Omega}_i, \overline{\Gamma}_i\right),$$

which we denote by S^-. In terms of the Jacobi polynomials (12) it is

$$S^- : (U, V, W) \mapsto (\overline{W}, \overline{V}, \overline{U}).$$

PROPOSITION 4.1. *The holomorphic involution $S^+ \circ S^- = S^- \circ S^+$ on $J(C_h;\infty^{\pm})$ is a translation on the half-period $\frac{1}{2}\Lambda_2$, where $\phi\left(\frac{1}{2}\Lambda_2\right) = 0 \in J(C_h)$ (see* (7), (9)).

The proof of the above Proposition will be given later in this section. If ϕ is the projection homomorphism defined in (7), then it implies

$$\phi \circ S^+ = \phi \circ S^- = S \circ \phi.$$

In other words the anti-holomorphic involutions S^+ and S^- "look alike" in the same way on the usual Jacobian $J(C_h)$ and differ in a half-period in the "vertical" direction with respect to ϕ on the generalized Jacobian $J(C_h;\infty^{\pm})$.

An important feature of S^+ is that the S^+-real part of the invariant level set T_h is preserved by the flow of (2). Indeed, changing the variables as

$$\begin{aligned} &\Omega_1 \longrightarrow i\Omega_1, && \Omega_2 \longrightarrow i\Omega_2, && \Omega_3 \longrightarrow \Omega_3, \\ &\Gamma_1 \longrightarrow i\Gamma_1, && \Gamma_2 \longrightarrow i\Gamma_2, && \Gamma_3 \longrightarrow \Gamma_3, \end{aligned}$$

we obtain a new system

$$\begin{aligned} &\dot{\Omega}_1 = -m\Omega_2\Omega_3 - \Gamma_2, && \dot{\Gamma}_1 = \Gamma_2\Omega_3 - \Gamma_3\Omega_2, \\ &\dot{\Omega}_2 = m\Omega_3\Omega_1 + \Gamma_1, && \dot{\Gamma}_2 = \Gamma_3\Omega_1 - \Gamma_1\Omega_3, \\ &\dot{\Omega}_3 = 0, && \dot{\Gamma}_3 = \Gamma_2\Omega_1 - \Gamma_1\Omega_2, \end{aligned} \tag{60}$$

with first integrals

$$H_1 = -\Gamma_1^2 - \Gamma_2^2 + \Gamma_3^2, \qquad H_2 = -\Omega_1\Gamma_1 - \Omega_2\Gamma_2 + (1+m)\Omega_3\Gamma_3,$$
$$H_3 = \tfrac{1}{2}\left(-\Omega_1^2 - \Omega_2^2 + (1+m)\Omega_3^2\right) - \Gamma_3, \qquad H_4 = \Omega_3.$$

The anti-holomorphic involution S^+ in these coordinates is given again by the complex conjugation.

THEOREM 4.2. *In each of the three connected subdomains of the complement to the discriminant locus of $f(\lambda)$ the topological type of the real part of the algebraic varieties $\big(J(C_h;\infty^\pm), S^\pm\big)$ and $(T_h, S^\pm)$ is one and the same and is given in the following table, where $T^2 = S^1 \times S^1$.*

roots of $f(\lambda)$	*no real roots*	*two real roots*	*four real roots*
real part of $\big(J(C_h;\infty^\pm), S^+\big)$	T^2	T^2	$T^2 \times (\mathbf{Z}/2)$
real part of $\big(J(C_h;\infty^\pm), S^-\big)$	T^2	$\varnothing$	$\varnothing$
real part of (T_h, S^+)	$S^1 \times \mathbf{R}$	$S^1 \times \mathbf{R}$	$T^2 \cup (S^1 \times \mathbf{R})$
real part of (T_h, S^-)	T^2	$\varnothing$	$\varnothing$

REMARK. It is easy to check that when the real invariant level set $T_h^{\mathbf{R}}$ of the Lagrange top is non-empty, then the polynomial $f(\lambda)$ has no real roots. If we do not use the generalized Jacobian $J(C_h;\infty^\pm)$, then it might be difficult to understand the relation between $T_h^{\mathbf{R}}$ (which has one connected component), $C_h^{\mathbf{R}}$ (which is empty) and $J(C_h)^{\mathbf{R}}$ (which has two connected components) (cf. [2], [3, p. 37]).

Proof of Proposition 4.1. We have $S^+ \circ S^-\colon (U,V,W) \mapsto (W,-V,U)$. The involution $(U,V,W) \mapsto (U,-V,W)$ is obviously induced by the elliptic involution $i\colon (\lambda,\mu) \mapsto (\lambda,-\mu)$ on C_h so it is a reflexion. This means that if a fixed point of i is taken for origin in $J(C_h;\infty^\pm)$ then $i = -$identity. It remains to prove that $j\colon (U,V,W) \mapsto (W,V,U)$ is a reflexion too. The involution j has the following simple geometrical interpretation. Let P_1, P_2 be two generic points in the (λ,μ) plane and lying on the affine curve $\check{C}_h = \{\mu^2 = f(\lambda)\}$. If $\{\mu = V(\lambda)\}$ is the straight line through P_1 and P_2 then it intersects C_h in four points P_1, P_2, P_3, P_4 and then $j(P_1+P_2) = P_3 + P_4$. Indeed, if the zero divisor of the Jacobi polynomial $U(\lambda)$ on C_h is $P_1 + P_2 + i(P_1) + i(P_2)$, then by (13) the zero divisor of $W(\lambda)$ is $P_3 + P_4 + i(P_3) + i(P_4)$ and the involution $P_1 + P_2 \mapsto P_3 + P_4$ amounts to exchanging the roots of $U(\lambda)$ and $V(\lambda)$.

Let W_i, $i = 1,\dots,4$ be the Weierstrass points on C_h. Then

$$\left(\frac{\mu - V(\lambda)}{\mu}\right) = \sum_{i=1}^{4} P_i - \sum_{i=1}^{4} W_i, \qquad \frac{\mu - V(\lambda)}{\mu} \overset{\infty}{\sim} 1$$

and hence on $J(C_h;\infty^\pm) \sim \mathrm{Div}^0(\check{C}_h)/\overset{m}{\sim}$ we have $P_1 + P_2 = -P_3 - P_4 +$ constant. This implies that j is a reflexion. Thus we have proved that $S^+ \circ S^-$

is a translation $(S^+ \circ S^-)(z) = z + a$. Finally, a is easily computed. We have $i(W_k) = W_k$, $j(W_1 + W_2) = W_3 + W_4$ and hence $a \overset{m}{\sim} W_1 + W_2 - W_3 - W_4$. Further if λ_1, λ_2 are zeros of $f(\lambda)$, then $(g) = W_1 + W_2 - W_3 - W_4$, where $g(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)/\mu$. Moreover $g(\infty^\pm) = \pm 1$, $g^2(\infty^\pm) = 1$ and hence

$$W_1+W_2-W_3-W_4 \sim 0\,, \quad W_1+W_2-W_3-W_4 \overset{m}{\not\sim} 0, \quad 2(W_1+W_2-W_3-W_4) \overset{m}{\sim} 0\,.$$

This shows that a is a half-period and $\phi(a) = 0 \in J(C_h)$. □

Proof of Theorem 4.2. The proof will consist of two steps. First we determine the action of $S^\pm$ on $H_1(\check{C}_h, \mathbf{Z})$ and hence on the period lattice Λ. From that we deduce the first two lines of the table. Second, we determine the action of $S^\pm : D_\infty \mapsto D_\infty$ on the infinity divisor $D_\infty = \phi^{-1}(p) = \mathbf{C}^2/\Lambda_2 \sim C^*$ and then we use that

$$\text{real part of } (T_h, S^\pm) = \text{real part of } (J(C_h; \infty^\pm), S^\pm) - \text{real part of } D_\infty\,.$$

It is easier to determine the action of S^+ on Λ. Indeed, S^+ is induced by an anti-holomorphic involution on C_h, $S^+ : (\lambda, \mu) \mapsto (\overline{\lambda}, -\overline{\mu})$. Note that S^+ always has fixed points on $J(C_h; \infty^\pm)$: if W_1, W_2 are two Weierstrass points on C_h such that either $W_1 = \overline{W}_2$, or W_1 and W_2 are S^+-real, then $S^+(W_1 + W_2) = W_1 + W_2$. On the other hand S^- has fixed points only if $f(\lambda)$ has no real roots. Indeed, in this last case let W_i, $i = 1, \ldots, 4$, be the Weierstrass points of C_h where $W_1 = \overline{W}_2$, $W_3 = \overline{W}_4$. Then $j(W_1 + W_3) = W_2 + W_4$ (see the proof of Proposition 4.1) and hence $S^-(W_1 + W_3) = W_1 + W_3$. On the other hand if $U = \overline{W}$ and $V = \overline{V}$, then

$$V^2(\lambda) + U(\lambda)W(\lambda) = |V(\lambda)|^2 + |U(\lambda)|^2 = f(\lambda) > 0 \qquad \forall \lambda \in \mathbf{R},$$

and hence $f(\lambda)$ has no real roots.

Suppose first that $f(\lambda)$ has no real roots and let us choose a basis A_1, B_1, A_2 of $H_1(\check{C}_h, \mathbf{Z})$ as shown in Figure 2 and in Figure 3 overleaf.

Then $S^+(A_1) = A_1$, $S^+(A_2) = A_2$ and it is easily seen that $S^+(B_1) + B_1$ is homologous to A_2 on $H_1(\check{C}_h, \mathbf{Z})$. Thus in the basis A_1, A_2, B_1 the matrix of the involution $S^+ : H_1(\check{C}_h, \mathbf{Z}) \to H_1(\check{C}_h, \mathbf{Z})$ takes the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

From this and the fact that $(J(C_h; \infty^\pm), S^+)$ is not empty we conclude that the real part of $(J(C_h; \infty^\pm), S^+)$ is a torus with generators the periods $\int_{B_1} \omega$ and

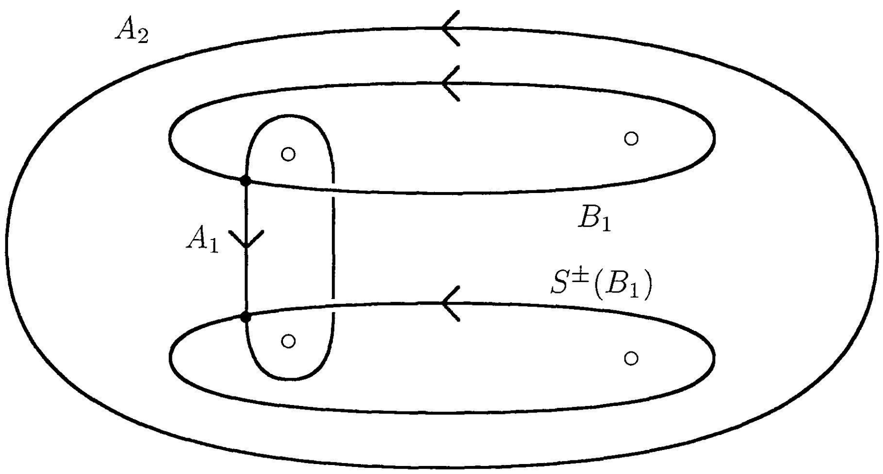

FIGURE 3
Projection of the cycles $A_1, B_1, A_2, S^{\pm}(B_1)$ on the λ–plane

$\int_{A_2}\omega$. On the other hand the real part of $\big(J(C_h;\infty^{\pm}), S^-\big)$ is also non-empty and $S^+ \circ S^-$ is a translation. We conclude that the real part of $\big(J(C_h;\infty^{\pm}), S^-\big)$ is just a translation of the real part of $\big(J(C_h;\infty^{\pm}), S^+\big)$ and in particular it is generated by the same periods.

In a similar way we find the real part of $\big(J(C_h;\infty^{\pm}), S^+\big)$ in the remaining cases. Note that in an appropriate $\mathbf{Z}$ basis of $H_1(\check{C}_h, \mathbf{Z})$ the matrix of the involution $S^{\pm}\colon H_1(\check{C}_h, \mathbf{Z}) \to H_1(\check{C}_h, \mathbf{Z})$ takes the same form if $f(\lambda)$ has two real roots, and it is of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

if $f(\lambda)$ has four real roots. This implies the first two lines of the table.

Let us determine now the real part of $(D_\infty, S^{\pm})$. As $D_\infty = \mathbf{C}^*/\Lambda_2$ then we have to compute $S^{\pm}(\Lambda_2)$. Note that, as the real invariant manifold T_h is compact, then (D_∞, S^-) is always empty. On the other hand (D_∞, S^+) is never empty. Indeed, if $S^+(\lambda,\mu) = (\overline{\lambda}, -\overline{\mu})$ then for $Q \in C_h$ the point $Q + S^+(Q)$ is S^+-real on $J(C_h;\infty^{\pm})$. As $S^+(\infty^+) = \infty^-$ we see that an S^+-real point of $\phi^{-1}(p)$ is obtained by taking the limit $Q \mapsto \infty^+$ in $S^+(Q) + Q$ along an appropriate real analytic curve on $\check{C}_h$. Finally, from the computation of the action of S^+ on Λ we get $S^+(\Lambda_2) = \Lambda_2$ which shows that the S^+-real part of $(\phi^{-1}(p), S^+)$ is always a circle $\mathbf{R}/\Lambda_2$. This gives the last two lines in the table. $\square$