

# Appendix: Proof of Proposition 3.1

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## APPENDIX: PROOF OF PROPOSITION 3.1

We start with some lemmas.

LEMMA A.1. *Let  $A$  be a commutative ring with identity on which a finite group  $G$  acts, let  $A^G$  be the invariant subring, and let  $B$  be a flat  $A^G$ -algebra. Then  $G$  acts on  $B \otimes_{A^G} A$  through its action on the second factor and the invariant subring of this action is  $B$ .*

*Proof.* We have an exact sequence

$$0 \rightarrow A^G \rightarrow A \rightarrow \prod_{g \in G} A,$$

where  $\prod_{g \in G} A$  is the direct product of  $|G|$  copies of  $A$ , and  $A \rightarrow \prod_{g \in G} A$  is defined by  $a \mapsto (ga - a)$ . Since  $B$  is a flat  $A^G$ -algebra, the tensor product of  $B$  with the above sequence remains exact, that is, the sequence

$$0 \rightarrow B \rightarrow B \otimes_{A^G} A \rightarrow \prod_{g \in G} B \otimes_{A^G} A$$

is exact. Hence  $B = (B \otimes_{A^G} A)^G$ .

Let  $A$  be a finitely generated  $k$ -algebra on which a finite group  $G$  acts. Then  $A$  is finite over  $A^G$ . For every prime ideal  $\mathfrak{q}$  of  $A^G$ , let  $\mathfrak{p}_1, \dots, \mathfrak{p}_n$  be all the prime ideals of  $A$  lying over  $\mathfrak{q}$ . It is known that  $G$  acts transitively on  $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ . Fix a  $\mathfrak{p} \in \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ . Let  $G_d = \{g \in G \mid g\mathfrak{p} = \mathfrak{p}\}$  be the decomposition group at  $\mathfrak{p}$ .

LEMMA A.2. *Notation as above. Let  $\widehat{A^G_{\mathfrak{q}}}$  be the completion of the local ring  $A^G_{\mathfrak{q}}$  and let  $\widehat{A_{\mathfrak{p}}}$  be the completion of the local ring  $A_{\mathfrak{p}}$ . Then  $G_d$  acts on  $\widehat{A_{\mathfrak{p}}}$  and  $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G_{\mathfrak{q}}}$ .*

*Proof.* Since  $A^G_{\mathfrak{q}}$  is a flat  $A^G$ -algebra, we have  $(A^G_{\mathfrak{q}} \otimes_{A^G} A)^G = A^G_{\mathfrak{q}}$  by Lemma A.1. Replacing  $A$  by  $A^G_{\mathfrak{q}} \otimes_{A^G} A$  if necessary, we may thus assume that  $A^G$  is a local ring and  $\mathfrak{q}$  is the maximal ideal of  $A^G$ .

Let  $\widehat{A}$  be the completion of  $A$  with respect to the  $\mathfrak{q}A$ -adic topology. Since  $A$  is a finite  $A^G$ -algebra, we have  $\widehat{A} = \widehat{A^G} \otimes_{A^G} A$ . On the other hand, we have  $\widehat{A} = \prod_i \widehat{A_{\mathfrak{p}_i}}$ . Since  $\widehat{A^G}$  is a flat  $A^G$ -algebra, we have  $\widehat{A^G} = (\widehat{A^G} \otimes_{A^G} A)^G$  by Lemma A.1. So we have  $\widehat{A^G} = (\prod_i \widehat{A_{\mathfrak{p}_i}})^G$ . Obviously  $(\prod_i \widehat{A_{\mathfrak{p}_i}})^G = (\widehat{A_{\mathfrak{p}}})^{G_d}$ . Therefore  $(\widehat{A_{\mathfrak{p}}})^{G_d} = \widehat{A^G_{\mathfrak{q}}}$ .

LEMMA A.3. *Let  $A$  be a noetherian local ring, let  $I_i$  ( $i = 1, \dots, n$ ) be some ideals of  $A$ , and let  $K_i$  be the kernel of the canonical homomorphism  $\widehat{A} \rightarrow \widehat{A/I_i}$ . If  $I = I_1 \cdots I_n$ , then the kernel of  $\widehat{A} \rightarrow \widehat{A/I}$  is  $K_1 \cdots K_n$ .*

*Proof.* Since  $A$  is noetherian, we have  $\ker(\widehat{A} \rightarrow \widehat{A/I_i}) = \widehat{I_i} = I_i \widehat{A}$ , that is  $K_i = I_i \widehat{A}$ . Similarly we have  $\ker(\widehat{A} \rightarrow \widehat{A/I}) = I \widehat{A} = I_1 \cdots I_n \widehat{A}$ . So  $\ker(\widehat{A} \rightarrow \widehat{A/I}) = K_1 \cdots K_n$ .

Let  $T$  be a  $k$ -scheme. Consider the Cartesian square

$$\begin{array}{ccc} X_m \times T & \longrightarrow & X_m \\ q \downarrow & & \downarrow \\ T & \longrightarrow & \text{spec}(k) . \end{array}$$

We have the following

LEMMA A.4. *Let  $s: T \rightarrow X_m \times T$  be a section of  $q$ . Then  $s$  is a closed immersion and the closed subscheme  $D$  defined by  $s$  is a relative effective Cartier divisor on  $X_m \times T/T$ .*

*Proof.* Since  $qs = \text{id}$  is a closed immersion and since  $q$  is separated,  $s$  is also a closed immersion. The closed subscheme  $D$  defined by  $s$  is flat because  $qs = \text{id}$ . Let  $\mathcal{I}$  be the sheaf of  $\mathcal{O}_{X_m \times T}$ -ideals defining  $D$ . We have an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{X_m \times T} \rightarrow \mathcal{O}_D \rightarrow 0 .$$

For any  $t \in T$ , since  $\mathcal{O}_D$  is  $\mathcal{O}_T$  flat, the following sequence is exact:

$$0 \rightarrow \mathcal{I} \otimes_{\mathcal{O}_T} k(t) \rightarrow \mathcal{O}_{X_m \times T} \otimes_{\mathcal{O}_T} k(t) \rightarrow \mathcal{O}_{D_t} \rightarrow 0 ,$$

where  $D_t$  is the fiber of  $D \rightarrow T$  at  $t$ . Hence  $\mathcal{I} \otimes_{\mathcal{O}_T} k(t)$  is the ideal defining the closed subscheme  $D_t$  of  $q^{-1}(t) \cong X_m$ . Obviously  $D_t$  defines a divisor of  $X_m$ . So for every point  $x \in q^{-1}(t)$ , the ideal  $\mathcal{I}_x \otimes_{\mathcal{O}_T} k(t)$  of  $\mathcal{O}_{X_m \times T, x} \otimes_{\mathcal{O}_T, t} k(t)$  is generated by an element which is not a zero divisor. By Nakayama's lemma, the ideal  $\mathcal{I}_x$  of  $\mathcal{O}_{X_m \times T, x}$  is generated by one element whose image in  $\mathcal{O}_{X_m \times T, x} \otimes_{\mathcal{O}_T, t} k(t)$  is not a zero divisor. By Lemma 2.3,  $D$  is a relative effective Cartier divisor.

Consider the sections

$$s_i: (X - S)^n \rightarrow X_m \times (X - S)^n, \quad (P_1, \dots, P_n) \mapsto (P_i, P_1, \dots, P_n).$$

Denote the relative effective Cartier divisors defined by  $s_i$  also by  $s_i$ , and let  $D = s_1 + \dots + s_n$ . The relative effective Cartier divisor  $D$  can also be regarded as a closed subscheme of  $X_m \times (X - S)^n$ . The  $n$ -th symmetric group  $\mathfrak{S}_n$  acts on  $(X - S)^n$  by permuting the factors. It acts on  $X_m \times (X - S)^n$  through its action on the second factor. Obviously  $D$  is stable under this action. Let  $\mathcal{D}$  be the quotient of  $D$  by  $\mathfrak{S}_n$ .

PROPOSITION A.5.

- (a) *The quotient of  $X_m \times (X - S)^n$  by  $\mathfrak{S}_n$  is  $X_m \times (X - S)^{(n)}$ .*
- (b) *The closed immersion  $D \rightarrow X_m \times (X - S)^n$  induces a closed immersion  $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$  and  $\mathcal{D}$  is a relative effective Cartier divisor on  $(X_m \times (X - S)^{(n)})/(X - S)^{(n)}$ . Moreover  $D$  is the pull-back of  $\mathcal{D}$ .*

*Proof.* (a) We have a Cartesian square

$$\begin{array}{ccc} X_m \times (X - S)^n & \longrightarrow & X_m \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ (X - S)^n & \longrightarrow & (X - S)^{(n)}. \end{array}$$

The morphism  $X_m \times (X - S)^{(n)} \rightarrow (X - S)^{(n)}$  is flat since it is obtained from the flat morphism  $X_m \rightarrow \text{spec}(k)$  through the base extension  $(X - S)^{(n)} \rightarrow \text{spec}(k)$ . Our assertion then follows directly from Lemma A.1.

(b) Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_m \times (X - S)^n & \longrightarrow & X_m \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ (X - S)^n & \longrightarrow & (X - S)^{(n)}. \end{array}$$

One can easily show that  $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$  is a finite morphism and induces a homeomorphism of  $\mathcal{D}$  with a closed subset of  $X_m \times (X - S)^{(n)}$ . We are going to show that for any point  $y \in \mathcal{D}$ , the homomorphism  $\mathcal{O}_{X_m \times (X - S)^{(n)}, y} \rightarrow \mathcal{O}_{\mathcal{D}, y}$  is surjective and the homomorphism  $\mathcal{O}_{(X - S)^{(n)}, t} \rightarrow \mathcal{O}_{\mathcal{D}, y}$  is flat, where  $t$  is the image of  $y$  in  $(X - S)^{(n)}$ . If this is done, then  $\mathcal{D} \rightarrow X_m \times (X - S)^{(n)}$

is a closed immersion and  $\mathcal{D} \rightarrow (X - S)^{(n)}$  is flat. Obviously the fibers of  $\mathcal{D} \rightarrow (X - S)^{(n)}$  are effective divisors. As in the proof of Lemma A.4, one can then use Nakayama's lemma and Lemma 2.3 to show that  $\mathcal{D}$  is a relative effective Cartier divisor.

One can show that

$$\widehat{\mathcal{O}}_{\mathcal{D},y} \cong \mathcal{O}_{\mathcal{D},y} \otimes_{\mathcal{O}_{X_m \times (X-S)^{(n)},y}} \widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y}.$$

Note that  $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y}$  is a faithfully flat  $\mathcal{O}_{X_m \times (X-S)^{(n)},y}$ -algebra. Thus to show that  $\mathcal{O}_{X_m \times (X-S)^{(n)},y} \rightarrow \mathcal{O}_{\mathcal{D},y}$  is surjective, it is enough to show that  $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)},y} \rightarrow \widehat{\mathcal{O}}_{\mathcal{D},y}$  is surjective; and to show that  $\mathcal{O}_{(X-S)^{(n)},t} \rightarrow \mathcal{O}_{\mathcal{D},y}$  is flat, it is enough to show that  $\widehat{\mathcal{O}}_{(X-S)^{(n)},t} \rightarrow \widehat{\mathcal{O}}_{\mathcal{D},y}$  is flat.

Assume  $t = n_1 P_1 + \cdots + n_l P_l \in (X - S)^{(n)}$ , where the  $P_i$  are distinct points of  $X - S$ ,  $n_i > 0$  and  $\sum_i n_i = n$ . Then  $y = (P_{i_0}, t) \in X_m \times (X - S)^{(n)}$  for some  $i_0 \in \{1, \dots, l\}$ . Let  $t' = (P_1, \dots, P_1, \dots, P_l, \dots, P_l) \in (X - S)^n$ , where the first  $n_1$  components of  $t'$  are  $P_1, \dots$ , and the last  $n_l$  components are  $P_l$ . The point  $t'$  is a point in  $(X - S)^n$  lying over  $t \in (X - S)^{(n)}$ . Let  $y'$  be the point  $(P_{i_0}, t')$  in  $X_m \times (X - S)^n$ . It lies over  $y$ . Note that  $y'$  is also a point in  $D$ . With respect to the actions of  $\mathfrak{S}_n$  on  $(X - S)^n$ , on  $X_m \times (X - S)^n$ , and on  $D$ , the decomposition groups at  $t' \in (X - S)^n$ , at  $y' \in X_m \times (X - S)^n$ , and at  $y' \in D$  are all  $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$ . We have

$$\widehat{\mathcal{O}}_{(X-S)^n,t} \cong k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

and the decomposition group  $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$  acts on  $\widehat{\mathcal{O}}_{(X-S)^n,t}$  by permuting  $x_{i1}, \dots, x_{in_i}$  for each  $i$ . We have

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

and the decomposition group  $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_l}$  acts on  $\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'}$  by fixing  $x$  and permuting  $x_{i1}, \dots, x_{in_i}$  for each  $i$ .

For each  $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \dots, n_1 + \cdots + n_{i_0}\}$ , the section

$$s_i: (X - S)^n \rightarrow X_m \times (X - S)^n, \quad (P_1, \dots, P_n) \mapsto (P_i, P_1, \dots, P_n)$$

induces a homomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \rightarrow \widehat{\mathcal{O}}_{(X-S)^n, t'}.$$

Through the isomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]]$$

and the isomorphism

$$\widehat{\mathcal{O}}_{(X-S)^n, t'} \cong k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

this homomorphism induced by  $s_i$  is

$$k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]] \rightarrow k[[x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]],$$

$$x \mapsto x_{i_0j}, \quad x_{\alpha\beta} \mapsto x_{\alpha\beta} \quad (\alpha = 1, \dots, l, \beta = 1, \dots, n_\alpha),$$

where  $j \in \{1, \dots, n_{i_0}\}$  is uniquely determined by  $n_1 + \dots + n_{i_0-1} + j = i$ . The kernel of this homomorphism is the ideal  $(x - x_{i_0j})$ . By Lemma A.3, the kernel of the homomorphism  $\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \rightarrow \widehat{\mathcal{O}}_{D, y'}$  is identified with the ideal  $\left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right)$  through the isomorphism

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^n, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]].$$

Hence

$$\widehat{\mathcal{O}}_{D, y'} \cong k[[x, x_{11}, \dots, x_{1n_1}, \dots, x_{l1}, \dots, x_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right),$$

and the decomposition group  $\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_l}$  acts on  $\widehat{\mathcal{O}}_{D, y'}$  by fixing  $x$  and permuting  $x_{i1}, \dots, x_{in_i}$  for each  $i$ . Let  $\sigma_{i1}, \dots, \sigma_{in_i}$  be the elementary symmetric functions in  $x_{i1}, \dots, x_{in_i}$ . By Lemma A.2, we have

$$\widehat{\mathcal{O}}_{D, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left(\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})\right),$$

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)}, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]],$$

$$\widehat{\mathcal{O}}_{(X-S)^{(n)}, t} \cong k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

Now it is easy to see that  $\widehat{\mathcal{O}}_{X_m \times (X-S)^{(n)}, y} \rightarrow \widehat{\mathcal{O}}_{D, y}$  is surjective and  $\widehat{\mathcal{O}}_{(X-S)^{(n)}, t} \rightarrow \widehat{\mathcal{O}}_{D, y}$  is flat. This proves  $\mathcal{D}$  is a relative effective Cartier divisor. We also have

$$\widehat{\mathcal{O}}_{D, y'} = \widehat{\mathcal{O}}_{D, y} \widehat{\otimes}_{\widehat{\mathcal{O}}_{(X-S)^{(n)}, t}} \widehat{\mathcal{O}}_{(X-S)^n, t'}.$$

This implies that  $D = \mathcal{D} \times_{(X-S)^{(n)}} (X-S)^n$ , that is,  $D$  is the pull-back of  $\mathcal{D}$ . This completes the proof of the proposition.

The relative effective Cartier divisor  $\mathcal{D}$  is the universal relative effective Cartier divisor.

LEMMA A.6. *Let  $T$  be a  $k$ -scheme and let  $s_i: T \rightarrow X_m \times T$  ( $i = 1, \dots, n$ ) be some sections of the projection  $q: X_m \times T \rightarrow T$ . Assume the images of  $s_i$  lie in  $(X_m - Q) \times T$ . Then there is a unique morphism of schemes  $f: T \rightarrow (X - S)^{(n)}$  such that the pull-back by  $\text{id} \times f$  of the universal relative effective Cartier divisor  $\mathcal{D}$  to  $X_m \times T$  is  $s_1 + \dots + s_n$ .*

*Proof.* Let  $p: X_m \times T \rightarrow X_m$  be the projection. The morphisms  $ps_i: T \rightarrow X_m$  induce  $(ps_1, \dots, ps_n): T \rightarrow X_m^n$ . Since the images of  $s_i$  lie in  $(X_m - Q) \times T$ , we actually get a morphism  $(ps_1, \dots, ps_n): T \rightarrow (X - S)^n$ . Composing with the canonical morphism  $(X - S)^n \rightarrow (X - S)^{(n)}$ , we get  $f: T \rightarrow (X - S)^{(n)}$  so that the pull-back of  $\mathcal{D}$  by  $\text{id} \times f$  is  $s_1 + \dots + s_n$ . This proves the existence of  $f$ .

To prove the uniqueness of  $f$ , we first note that  $f: T \rightarrow (X - S)^{(n)}$  is uniquely determined as a map on the underlying topological space. Indeed, for every point  $t \in T$ ,  $f(t)$  is necessarily the point in  $(X - S)^{(n)}$  corresponding to the effective divisor  $(s_1 + \cdots + s_n)_t$  on  $q^{-1}(t) = X_m$ . To prove  $f$  is unique as a morphism of schemes, it is enough to prove that the homomorphism on local rings  $\mathcal{O}_{(X-S)^{(n)}, f(t)} \rightarrow \mathcal{O}_{T, t}$  induced by  $f$  is uniquely determined. It suffices to prove that  $\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \rightarrow \widehat{\mathcal{O}}_{T, t}$  is uniquely determined.

Consider the commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ X_{\mathfrak{m}} \times T & \longrightarrow & X_{\mathfrak{m}} \times (X - S)^{(n)} \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & (X - S)^{(n)}, \end{array}$$

where  $D$  is the closed subscheme of  $X_m \times T$  corresponding to the divisor  $s_1 + \cdots + s_n$ . Let  $A = \widehat{\mathcal{O}}_{T,t}$ , let  $z \in D$  be a point lying over  $t \in T$ , and let  $y \in \mathcal{D}$  be the image of  $z$ . We have  $\widehat{\mathcal{O}}_{X_m \times T, z} \cong A[[x]]$ .

Without loss of generality, assume

$$\begin{aligned} ps_1(t) &= \cdots = ps_{n_1}(t) = P_1, \\ ps_{n_1+1}(t) &= \cdots = ps_{n_1+n_2}(t) = P_2, \end{aligned}$$

$$ps_{n_1+\dots+n_{l-1}+1}(t) = \dots = ps_{n_1+\dots+n_l}(t) = P_l,$$

where  $n_i > 0$  ( $i = 1, \dots, l$ ),  $n_1 + \dots + n_l = n$ , and the  $P_i$  are distinct points in  $X - S$ . Then we have  $z = (P_{i_0}, t) \in X_m \times T$  for some  $i_0 \in \{1, \dots, l\}$ .

For each  $i \in \{n_1 + \cdots + n_{i_0-1} + 1, \dots, n_1 + \cdots + n_{i_0}\}$ , the section  $s_i$  induces a homomorphism  $\widehat{\mathcal{O}}_{X_m \times T, z} \rightarrow \widehat{\mathcal{O}}_{T, t}$ , i.e.,  $A[[x]] \rightarrow A$ . Denote the image of  $x$  under this homomorphism by  $a_{i_0 j}$ , where  $j \in \{1, \dots, n_{i_0}\}$  is uniquely determined by  $n_1 + \cdots + n_{i_0-1} + j = i$ . Then by Lemma A.3, we have

$$\widehat{\mathcal{O}}_{D, z} \cong A[[x]] / \left( \prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right).$$

Keep the notations in the proof of Proposition A.5. We have

$$\widehat{\mathcal{O}}_{\mathcal{D}, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left( \prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right).$$

$$\widehat{\mathcal{O}}_{X_m \times (X-S)^{(m)}, y} \cong k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

$$\widehat{\mathcal{O}}_{(X-S)^{(m)}, f(t)} \cong k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]].$$

We have a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{D, z} & \longleftarrow & \widehat{\mathcal{O}}_{\mathcal{D}, y} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{O}}_{X_m \times T, z} & \longleftarrow & \widehat{\mathcal{O}}_{X_m \times (X-S)^{(m)}, y} \\ \uparrow & & \uparrow \\ \widehat{\mathcal{O}}_{T, t} & \longleftarrow & \widehat{\mathcal{O}}_{(X-S)^{(m)}, f(t)} \end{array}$$

It is isomorphic to

$$\begin{array}{ccc} A[[x]] / \left( \prod_{j=1}^{n_{i_0}} (x - a_{i_0 j}) \right) & \longleftarrow & k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] / \left( \prod_{j=1}^{n_{i_0}} (x - x_{i_0 j}) \right) \\ \uparrow & & \uparrow \\ A[[x]] & \longleftarrow & k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \\ \uparrow & & \uparrow \\ A & \longleftarrow & k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]]. \end{array}$$



In order for this last diagram to commute, it is necessary that  $\prod_{j=1}^{n_{i_0}} (x - x_{i_0j})$  be mapped to  $\prod_{j=1}^{n_{i_0}} (x - a_{i_0j})$  under the homomorphism

$$k[[x, \sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \rightarrow A[[x]].$$

So the image of  $\sigma_{i_0j}$  under the homomorphism

$$k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \rightarrow A$$

is necessarily the value at  $(a_{i_01}, \dots, a_{i_0n_{i_0}})$  of  $\sigma_{i_0j}$  considered as a function on  $A^{n_{i_0}}$ . We see that this is true for any indices  $i_0$  and  $j$  if we let  $z$  go over the points in  $D$  above  $t$ . Therefore the homomorphism  $k[[\sigma_{11}, \dots, \sigma_{1n_1}, \dots, \sigma_{l1}, \dots, \sigma_{ln_l}]] \rightarrow A$  is uniquely determined, that is, the homomorphism  $\widehat{\mathcal{O}}_{(X-S)^{(n)}, f(t)} \rightarrow \widehat{\mathcal{O}}_{T,t}$  is uniquely determined. This concludes the proof of the lemma.

**LEMMA A.7.** *Let  $T$  be a  $k$ -scheme and let  $D$  be a relative effective Cartier divisor on  $(X_m \times T)/T$  supported on  $(X_m - Q) \times T$  with degree  $n$ . Then there exist a flat morphism  $T' \rightarrow T$  and sections  $s_i: T' \rightarrow X_m \times T'$  ( $i = 1, \dots, n$ ) of the projection  $X_m \times T' \rightarrow T'$  such that the pull-back of  $D$  to  $X_m \times T'$  is equal to  $s_1 + \dots + s_n$ .*

*Proof.* By the definition of relative effective Cartier divisors,  $D$  is flat over  $T$ . On the other hand,  $D \rightarrow T$  is proper and has finite fibers. So  $D$  is finite over  $T$  by [EGA] III, §4.4.2. Take  $T_1 = D$ . Then we have a finite flat morphism  $T_1 \rightarrow T$ . Consider the commutative diagram

$$\begin{array}{ccccc} D \times_T T_1 & \xrightarrow{p'} & D & & \\ i' \downarrow & & i \downarrow & & \\ X_m \times T_1 & \xrightarrow{p} & X_m \times T & \longrightarrow & X_m \\ q' \downarrow & & q \downarrow & & \downarrow \\ D = T_1 & \xrightarrow{qi} & T & \longrightarrow & \text{spec}(k). \end{array}$$

Let  $\Delta: D \rightarrow D \times_T D = D \times_T T_1$  be the diagonal map. It is a closed immersion since the morphism  $qi$  is separated. Take  $s_1 = i'\Delta$ . This is a section of  $q'$ . Hence it defines a relative effective Cartier divisor on  $(X_m \times T_1)/T_1$ . The pull-back  $D_1$  of the relative effective Cartier divisor  $D$  to  $X_m \times T_1$  is the closed subscheme defined by  $i'$ . Let  $\mathcal{I}_{D_1}$  and  $\mathcal{I}_s$  be the ideal sheaves of the

closed subschemes defined by  $i'$  and  $s_1$ , respectively. Since  $s_1$  factors through  $i'$ , we have  $\mathcal{I}_{D_1} \subset \mathcal{I}_s$ . Hence  $D_1 - s$  is a relative effective Cartier divisor on  $(X_m \times T_1)/T_1$  by Lemma 2.2 (b), that is, there exists a relative effective Cartier divisor  $D_1'$  such that  $D_1 = s_1 + D_1'$ . Now we take  $T_2 = D_1'$ . We then have a finite flat morphism  $T_2 \rightarrow T_1$ , a section  $s_2: T_2 \rightarrow X_m \times T_2$  of the projection  $X_m \times T_2 \rightarrow T_2$ , and a relative effective Cartier divisor  $D_2'$  on  $(X_m \times T_2)/T_2$  such that the pull-back of  $D_1'$  to  $X_m \times T_2$  is equal to  $s_2 + D_2'$ . Then we take  $T_3 = D_2'$ , .... In this way we get finite flat morphisms  $T_i \rightarrow T_{i-1}$  ( $i = 1, \dots, n$ ), sections  $s_i: T_i \rightarrow X_m \times T_i$ , such that the pull-back of  $D$  to  $X_m \times T_n$  is equal to  $s_1 + \dots + s_n$ , where the  $s_i$  denote the relative effective Cartier divisors on  $(X_m \times T_n)/T_n$  induced by the sections  $s_i$ . This proves our lemma.

Finally we are ready to prove Proposition 3.1.

*Proof of Proposition 3.1.* By Lemma A.7, there exist a finite flat morphism  $\pi: T' \rightarrow T$  and sections  $s_i: T' \rightarrow X_m \times T'$  ( $i = 1, \dots, n$ ) of the projection  $X_m \times T' \rightarrow T'$  such that the pull-back  $\pi^*D$  of  $D$  to  $X_m \times T'$  is equal to  $s_1 + \dots + s_n$ . By Lemma A.6, there exists a unique morphism of schemes  $f': T' \rightarrow (X - S)^{(n)}$  such that the pull-back  $f'^*\mathcal{D}$  of the universal relative effective Cartier divisor  $\mathcal{D}$  to  $X_m \times T'$  is  $s_1 + \dots + s_n$ . Let  $p_1, p_2: T' \times_T T' \rightarrow T'$  be the projections. We have

$$(f'p_1)^*(\mathcal{D}) = p_1^*f'^*\mathcal{D} = p_1^*(s_1 + \dots + s_n) = p_1^*\pi^*D = p_2^*\pi^*D = \dots = (f'p_2)^*(\mathcal{D}).$$

that is,  $(f'p_1)^*(\mathcal{D}) = (f'p_2)^*(\mathcal{D})$ . By Lemma A.6 we have  $f'p_1 = f'p_2$ . By the theory of descent, ([SGA 1] VIII, Theorem 5.2), there exists a unique morphism of schemes  $f: T \rightarrow (X_m - Q)^{(n)}$  such that  $f' = f\pi$ , and the pull-back of  $\mathcal{D}$  to  $X_m \times T$  is  $D$ .

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