4.4 Wreath products

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

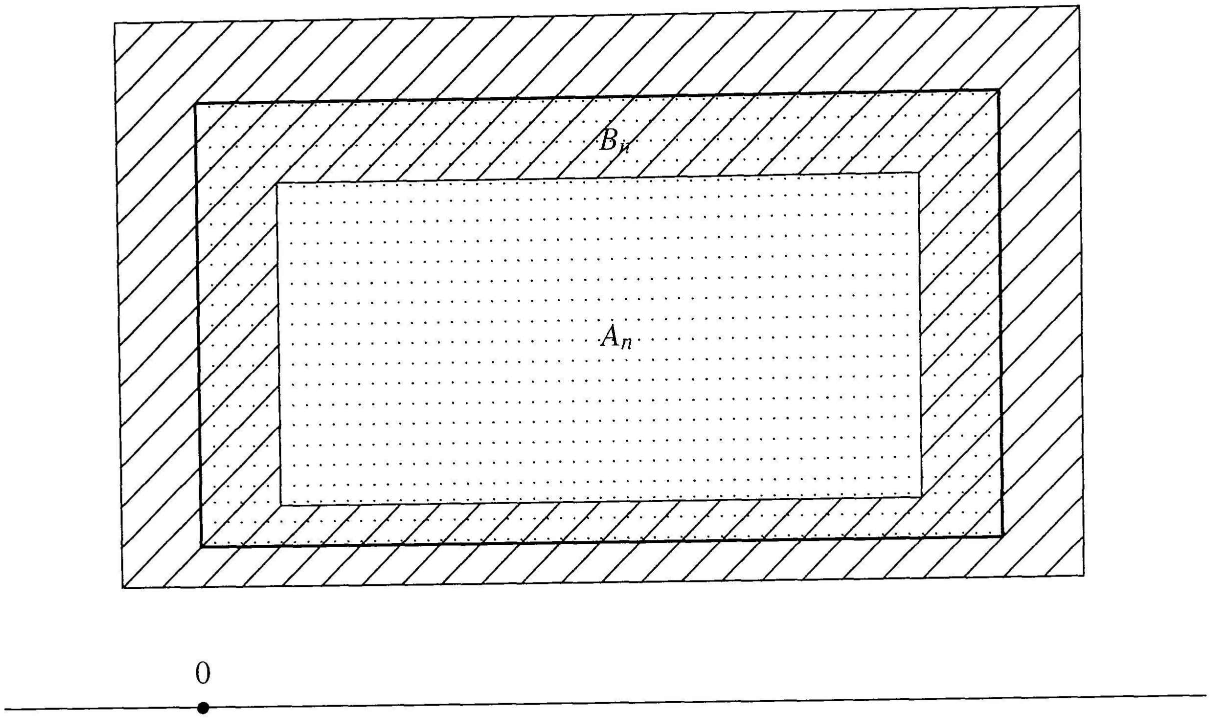

FIGURE 5
Sets A_n and B_n

$$\begin{aligned}B_n =&\{z \in H;\ -R \leq \mathrm{Re}(z) \leq R,\ e^R \geq \mathrm{Im}(z) \geq e^{-n-R}\}\\ &\cup \{z \in H;\ -R+n \leq \mathrm{Re}(z) \leq n+R,\ e^R \geq \mathrm{Im}(z) \geq e^{-n-R}\}\\ &\cup \{z \in H;\ -R \leq \mathrm{Re}(z) \leq n+R,\ e^R \geq \mathrm{Im}(z) \geq e^{-R}\}\\ &\cup \{z \in H;\ -R \leq \mathrm{Re}(z) \leq n+R,\ e^{-n+R} \geq \mathrm{Im}(z) \geq e^{-n-R}\}.\end{aligned}$$

One can see that

$$|B_n|_{f^2} \approx n, \quad |A_n|_{f^2} \approx n^2 .$$

This shows that $\{A_n\}_{n=1}^{\infty}$ is a generalized Følner sequence. Thus

$$||P||_{L^2(H,d_H z)\to L^2(H,d_H z)} = \int_{|z-i|=R} \sqrt{\mathrm{Im(z)}}\, dm_R(z).$$

4.4 WREATH PRODUCTS

Let G and F be finitely generated groups. We define the wreath product $G \wr F$ of these groups as follows. Elements of $G \wr F$ are couples (g, γ_1) where $g\colon F \to G$ is a function such that $g(\gamma)$ is different from the identity element id_G of G only for finitely many elements γ in F, and where γ_1 is an element of F. The multiplication in $G \wr F$ is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(\gamma) = g_1(\gamma)g_2(\gamma\gamma_1) \quad \text{for } \gamma \in F\,.$$

If S_G and S_F are generators of G and F respectively then

$$\{(g,\gamma)\ ;\ (g(F) = id_G,\ \gamma \in S_F) \text{ or } (g(F \setminus id_F) = id_G,\ g(id_F) \in S_G,\ \gamma = id_F)\}$$

is a generating subset for $G \wr F$.

Let μ and ν be symmetric, finitely supported probability measures on F and G respectively.

As there is a natural embedding of F and G into $G \wr F$, one can view the measures μ and ν as measures on $G \wr F$. More precisely:

$$\nu(g,\gamma) = \begin{cases} \nu(g(id_F)) & \text{if } \gamma = id_F \text{ and } g(F \setminus id_F) = id_G \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu(g,\gamma) = \begin{cases} \mu(\gamma) & \text{if } g(F) = id_G \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mu \star \nu \star \mu$ is a symmetric measure on $G \wr F$. Explicitly we have:

$$\mu \star \nu \star \mu(g,\gamma) = \begin{cases} \mu(\gamma(\gamma_0)^{-1})\mu(\gamma_0)\nu(g(\gamma_0)) & \text{if } g(F \setminus \gamma_0) = id_G \\ 0 & \text{otherwise}\,. \end{cases}$$

We want to prove:

THEOREM 7. *Let F and G be finitely generated groups. If F is amenable then the spectral radius of ν on G is the same as the spectral radius of $\mu \star \nu \star \mu$ on $G \wr F$.*

Proof. We will prove Theorem 7 by constructing on $G \wr F$ a positive function $\widetilde{f}$ which is an eigenfunction for the convolution by $\mu \star \nu \star \mu$ with eigenvalue $\|\nu\|_{l^2(G)\to l^2(G)}$ and for which there exists a generalized Følner sequence.

Let f be a positive eigenfunction for the operator which is a convolution on $l^2(G)$ by ν, corresponding to the eigenvalue $\|\nu\|$, i.e.

$$f \star \nu = \|\nu\| f\,. \tag{12}$$

We can normalize f so that

$$f(id_G) = 1\,. \tag{13}$$

By Theorem 3 (and the remark after its proof) there exists a sequence of finite subsets $A_n \subset G$, such that

$$\frac{\sum_{\gamma\in\partial A_n} f^2(\gamma)}{\sum_{\gamma\in A_n} f^2(\gamma)} \to_{n\to\infty} 0 .$$

As the group F is amenable there exists a sequence of finite subsets $B_n \subset F$, such that

$$\frac{\#\partial B_n}{\#B_n} \to_{n\to\infty} 0 .$$

For technical reasons let us choose the sequences B_n and A_n in such a way that

$$\frac{\#\partial B_n}{\#B_n} < \frac{1}{n} \quad \text{and} \quad \frac{\sum_{\gamma\in\partial A_n} f^2(\gamma)}{\sum_{\gamma\in A_n} f^2(\gamma)} < \frac{1}{n(\#B_n)} . \tag{14}$$

Now, on $G \wr F$ we define $\widetilde{f}$ as follows

$$\widetilde{f}(g,\gamma_1) = \prod_{\gamma\in F} f(g(\gamma)) .$$

The function $\widetilde{f}$ is well defined because by (13), $f(g(\gamma))$ is different from 1 only for finitely many $\gamma \in F$. This function is of course positive and does not depend on γ_1. From (12) one has

$$\widetilde{f} \star \mu \star \nu \star \mu = \widetilde{f} \star \nu \star \mu = \|\nu\| \widetilde{f} \star \mu = \|\nu\| \widetilde{f} .$$

To complete the proof of Theorem 7 it is enough to construct a generalized Følner sequence $C_n \subset G \wr F$ for $\widetilde{f}$. We define C_n as follows:

$$C_n = \{(g,\gamma_1); \gamma_1 \in B_n,\ g(B_n) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\} .$$

LEMMA 5. *The sequence $C_n \subset G \wr F$ is a generalized Følner sequence for $\widetilde{f}$.*

Proof. Let us define sets D_n and ∂D_n as follows:

$$\begin{aligned} D_n &= \{g\colon F \to G;\ g(B_n) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\} ,\\ \partial D_n &= \{g\colon F \to G;\ \text{there exists } \gamma_0 \in B_n \text{ such that } g(\gamma_0) \in \partial A_n,\\ &\qquad g(B_n \setminus \gamma_0) \subset A_n,\ g^{-1}(G \setminus id_G) \subset B_n\} . \end{aligned}$$

Thus

$$\begin{aligned} C_n &= D_n \times B_n ,\\ \partial C_n &= (\partial D_n \times B_n) \cup (D_n \times \partial B_n) . \end{aligned}$$

We have then

$$\sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2 = \sum_{(g,\gamma_1)\in C_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2$$
$$= \sum_{(g,\gamma_1)\in D_n\times B_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2 = \#B_n \sum_{g\in D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2.$$

On the other hand

$$\sum_{(g,\gamma_1)\in\partial C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2 = \sum_{(g,\gamma_1)\in\partial C_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2$$
$$= \sum_{(g,\gamma_1)\subset(\partial D_n\times B_n)\cup(D_n\times\partial B_n)} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2$$
$$= \#\partial B_n \sum_{g\in D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2 + \#B_n \sum_{g\in\partial D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2$$
$$= \frac{\#\partial B_n}{\#B_n} \sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2$$
$$+ \frac{\sum_{g\in\partial D_n}\left(\prod_{\gamma\in F} f(g(\gamma))\right)^2}{\sum_{g\in D_n}\left(\prod_{\gamma\in F} f(g(\gamma))\right)^2} \sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2.$$

But

$$\sum_{g\in\partial D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2 = \#B_n \frac{\sum_{\alpha\in\partial A_n} f^2(\alpha)}{\sum_{\alpha\in A_n} f^2(\alpha)} \sum_{g\in D_n} \left(\prod_{\gamma\in F} f(g(\gamma))\right)^2.$$

Thus by (14)

$$\sum_{(g,\gamma_1)\in\partial C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2 = \left(\frac{\#\partial B_n}{\#B_n} + \#B_n \frac{\sum_{\alpha\in\partial A_n} f^2(\alpha)}{\sum_{\alpha\in A_n} f^2(\alpha)}\right) \sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2$$
$$\le \frac{2}{n} \sum_{(g,\gamma_1)\in C_n} \left(\widetilde{f}(g,\gamma_1)\right)^2,$$

which shows that C_n is a generalized Følner sequence for $\widetilde{f}$. $\square$

This ends the proof of Theorem 7.