2. A COUPLE OF FACTS FROM COHOMOLOGY

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COROLLARY 2. *Let Σ be a finite subset of places of $\mathbf{Q}$. Assume that*

(a) For all places $v \notin \Sigma$ the function $f \in K$ is a norm from $\mathbf{Q}_v L$ to $\mathbf{Q}_v(t)$.

(b) For all $v \in \Sigma$ there exist $a_v, b_{i,v} \in \mathbf{Q}_v$ with

$$f(a_v) = N(a_v, b_{1,v}, \ldots, b_{d,v}) \in \mathbf{Q}_v^* .$$

Then f is a norm from L.

(In § 5 we shall see that (a) is not sufficient in itself.) Colliot-Thélène has shown me a different proof of this corollary using the above-mentioned Faddeev exact sequence, actually removing the regularity assumption. The result reminds one of the work by Pourchet (see [Raj, Lemma 17.4]) and by Colliot-Thélène, Coray, Sansuc [CThCS, Prop. 1.3]. (For instance the last paper contains the proof that a *multiplicative* quadratic form over $k(t)$ represents f over $k(t)$ if and only if it represents f over $k_v(t)$ for all places v of k.)

The paper is organized as follows. In § 2 we shall recall a few basics from cohomology. In § 3 we shall prove the theorem and its corollaries. In § 4 we shall discuss a simple counterexample to an analogous result when $\mathrm{Gal}(L/K)$ is a four-group (similarly to the number-field case). In § 5 we shall discuss how the assumptions for Corollary 2 are equivalent for large p both to the solvability of congruences $f \equiv N(g) \pmod{p}$ and to the existence of solutions over the completion of $\mathbf{Q}_p L$ under the Gauss norm. Incidentally, we shall prove that if a representation of f by N exists at all with the $x_i \in k(t)$, then some representation will have the x_i's of degree bounded explicitly only in terms of $\deg f$ and genus and degree of $kL/k(t)$. This seems to have some interest in itself. These observations lead also to the construction of varieties satisfying the usual local-global principle. Finally, in § 6 we shall discuss how to find effectively a possible representation of f by N.

2. A COUPLE OF FACTS FROM COHOMOLOGY

Let G be a finite group acting on an abelian group M. For a function $\xi \colon G \to M$, $\sigma \mapsto \xi_\sigma$ we denote (the usual coboundary operator)

$$\partial(\xi_\sigma) = \partial(\xi) \colon G^2 \to M , \qquad (\sigma, \tau) \mapsto \xi_\sigma + \sigma(\xi_\tau) - \xi_{\sigma\tau} .$$

With this notation (but writing M multiplicatively) we now recall Hilbert's Theorem 90:

Let k_1/k be a finite Galois extension with group G and let $\xi\colon G \to k_1^*$ be a function satisfying $\partial(\xi) = 1$. Then there exists $\alpha \in k_1^*$ such that $\xi_\sigma = \alpha/\sigma(\alpha)$ for all $\sigma \in G$.

The usual proof (see e.g. [CF, Prop. 3, p. 124]) is simple and runs as follows: For $x \in k_1$ form the sum $\alpha = \sum_{\sigma\in G} \xi_\sigma \sigma(x)$. By a well-known elementary result of Artin, we may choose $x \in k_1$ such that $\alpha \neq 0$. A quick computation using the assumption on ξ then shows that α has the stated property.

An easy corollary (the original Hilbert's 90) is that, *if G is cyclic generated by g, then every element $a \in k_1^*$ such that $N_k^{k_1}(a) = 1$ is of the form $b/g(b)$ for some $b \in k_1^*$*. To derive this conclusion it suffices to apply the above statement to the function on G defined by $\xi_{g^m} = \prod_{i=0}^{m-1} g^i(a)$ (which is well defined).

In § 6 on effectiveness we shall need a simple result on *permutation modules* for the action of a finite group G. Such a module is simply a free abelian group on which G acts, which moreover has a $\mathbf{Z}$-basis permuted by G. We have:

Let M be a permutation module and let $\xi\colon G \to M$ satisfy $\partial(\xi) = 0$. Then there exists $m \in M$ such that $\xi_\sigma = m - \sigma(m)$ for all $\sigma \in G$.

We give a short argument for completeness. We may write M as a direct sum of permutation modules, each of which has a $\mathbf{Z}$-basis which is a G-orbit. It suffices to prove the claim for each direct factor. Write the mentioned basis as $\{g(b)\}$ for a certain $b \in M$ and g running through a set of representatives for G/H, H being the stabilizer of b.

We sum the equations $\xi_{\sigma\tau} = \xi_\sigma + \sigma(\xi_\tau)$ over $\tau \in G$. Letting n be the order of G and putting $\mu := \sum_{g\in G} \xi_g \in M$, we get

$$n\xi_\sigma = \mu - \sigma(\mu)\,.$$

Write $\mu = \sum_{g\in G/H} a_g g(b)$ for suitable $a_g \in \mathbf{Z}$. The displayed equation implies $\mu \equiv \sigma(\mu) \pmod{nM}$ for every $\sigma \in G$. This immediately gives the existence of $a \in \mathbf{Z}$ such that $a_g \equiv a \pmod n$ for all $g \in G/H$, so we write $a_g = a + nq_g$ where $q_g \in \mathbf{Z}$. Let $m := \sum_{G/H} q_g g(b) \in M$. Then $nm = \mu - a\sum_{G/H} g(b)$, where the last term is invariant by G. Hence $n\xi_\sigma = n(m - \sigma(m))$, whence $\xi_\sigma = m - \sigma(m)$, as required.