4. An example for the non-cyclic case

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$$N(r, b_1, \ldots, b_d) = f(r)\mu$$

where $\mu \in \mathbf{Q}_v$ is very close to 1; in fact $f(r)$ is near to $f(a_v)$, which is nonzero. By the previous remarks, μ^{-1} is in the image of $N(r, x_1, \ldots, x_d)$ on $\mathbf{Q}_v^d$, hence the same must be true for $f(r)$, by the basic multiplicative identity for N. In particular $f(r)$ will be a norm from $\mathbf{Q}_v(P_r)$ to $\mathbf{Q}_v$.

Let now S consist of the elements of $\mathbf{Q}$ which are not poles or zeros of f, which satisfy $[\mathbf{Q}(P_s) : \mathbf{Q}] = d$ and which are sufficiently close (in the mentioned sense) to a_v, for each $v \in \Sigma$. We have proved that $f(s)$ is a norm from $\mathbf{Q}_v(P_s)$, for all $s \in S$ and for all places v. By Hasse's theorem, $f(s)$ is a norm from $\mathbf{Q}(P_s)$, so $S \subset N_f$. On the other hand $S \cap \mathbf{Z}$ contains the complement of a thin set in an arithmetic progression, whence N_f cannot satisfy the conclusion of the Theorem (or of Corollary 1), as required. $\square$

4. AN EXAMPLE FOR THE NON-CYCLIC CASE

We show that assuming that L/K is cyclic is essential in the Theorem (as in the number-field case, as shown in [CF, Ex. 5]).

To describe a counterexample, define $L = \mathbf{Q}(t, \sqrt{4t+3}, \sqrt{4t+7})$, $f(t) = t^2$. We proceed to show that $\mathbf{N} \subset N_f$. We have to show that for all large integers n, n^2 is a norm from $L(n) := \mathbf{Q}(\sqrt{4n+3}, \sqrt{4n+7})$. By [CF, Ex. 5.1 and 5.2, p. 360] it is sufficient to show that the local degree $[L(n)_w : \mathbf{Q}_p]$ is 4 for some prime p. Observe that the Jacobi symbol $\left(\frac{4n+3}{4n+7}\right) = \left(\frac{-1}{4n+7}\right) = -1$. Hence there exists some prime p dividing $4n+7$ with an odd multiplicity and such that $\left(\frac{4n+3}{p}\right) = -1$. Then p ramifies in $L(n)$ and the residual degree is 2, proving the claim. Observe that the first conclusion of Corollary 1 does not hold for N_f.

On the other hand, t^2 is not a norm from L to K. Otherwise by [CF, Ex. 5.1] we could write t as the product of three norms from the three quadratic subfields of L. In other words we could write nontrivially

$$q^2(t)t = (a_1^2(t) - (4t+3)b_1^2(t))(a_2^2(t) - (4t+7)b_2^2(t))(a_3^2(t) - (4t+3)(4t+7)b_3^2(t)),$$

where $q, a_i, b_j \in \mathbf{Q}[t]$. We may suppose that a_i and b_i are coprime for each i, otherwise we can divide out a common factor. Now, putting $t = 0$ we get a contradiction.