Zeitschrift: L'Enseignement Mathématique

Band: 45 (1999)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON GROUPS ACTING ON NONPOSITIVELY CURVED CUBICAL COMPLEXES

Kapitel: Introduction

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DOI: https://doi.org/10.5169/seals-64441

L'Enseignement Mathématique, t. 45 (1999), p. 51–81

ON GROUPS ACTING ON NONPOSITIVELY CURVED CUBICAL COMPLEXES

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ABSTRACT. We study groups acting on simply connected cubical complexes of nonpositive curvature. Our main objectives are related actions on trees, the existence of free subgroups and the existence of homomorphisms onto free abelian groups.

INTRODUCTION

We study groups acting on simply connected cubical complexes of nonpositive curvature. Examples of such groups and spaces arise naturally from many constructions. Among them are graph products of groups and other groups acting on right-angled buildings, fundamental groups of hyperbolizations of polyhedra, of toric manifolds and of blow-ups of arrangements of hyperplanes, and many others (see [Da], [DJ1], [DJ2], [DJS] and Section 2 below). Roughly speaking, a *cubical complex* is a cell complex whose cells are cubes. As a definition of nonpositive curvature we use the comparison triangle condition $CAT(0)$ with respect to the natural *cubical metric* of a cubical complex (see Section 1 below for more details).

It turns out that groups acting on nonpositively curved cubical complexes share many properties with groups acting on trees and with infinite Coxeter groups. For example, if Γ is a group satisfying Property (T), then any automorphic action of Γ on a tree, a Coxeter complex, a Euclidean space or a hyperbolic space has a fixed point, see [HV], Chapter 6. The same result holds for actions of Γ on cubical complexes, a result recently proved by Niblo and Reeves, see [NR]. This result and our related results in [BS] are the source of our interest in cubical complexes.

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[1]) Partially supported by Sonderforschungsbereich 256, Universität Bonn

[2]) Partially supported by Max-Planck-Institut für Mathematik (Bonn), SFB256 (Bonn) and Polish Committee of Scientific Research, grant no. 1262/P03/95/08

For the results in this paper we require stronger assumptions on cubical complexes, related to finer results in [BS]. These additional requirements are still natural and are satisfied by many examples. First of all, we only consider *chamber complexes*. Our second and main requirement is that the complex X is *foldable*, that is, it admits a combinatorial map onto an n-cube, where $n = \dim X$. Since a folding of an n-dimensional cubical chamber complex X is unique up to an automorphism of the n-cube, any group Γ of automorphisms of X contains a finite index subgroup Γ' preserving a given (and hence any) folding of X. We refer the reader to Section 1 for definitions and basic facts.

We recall that a *Hadamard space* is a simply connected complete space of nonpositive curvature. The theory of Hadamard spaces is fundamental for the arguments in this paper. Isometries of Hadamard spaces fall into three classes according to the behaviour of their corresponding displacement function. If this function assumes its infimum, then the corresponding isometry is called *semisimple*, otherwise *parabolic*. The semisimple isometries fall into two subclasses, the *elliptic* ones which fix a point and the *axial* ones which translate a geodesic of the space.

Associated to a Hadamard space X is the *ideal boundary* $X(\infty)$ at infinity and the *closure* $\bar{X} = X \cup X(\infty)$. These objects generalize the corresponding objects for trees and the hyperbolic plane in an appropriate way. For details we refer to [Ba].

As we mentioned above, a group does not satisfy Property (T) if it acts without fixed points on a tree. In this sense, the result below gives a strengthening of the result of Niblo and Reeves.

THEOREM 1. *Let X be a simply connected foldable cubical chamber complex of nonpositive curvature, and let $\mathrm{Aut}_f(X)$ be the group of automorphisms of X preserving the foldings. Then we have:*

(1) *there are simplicial trees $\Lambda_1^*, \ldots, \Lambda_n^*$, $n = \dim X$, actions of $\mathrm{Aut}_f(X)$ on $\Lambda_1^*, \ldots, \Lambda_n^*$ and a biLipschitz embedding $r\colon X \to \Lambda_1^* \times \cdots \times \Lambda_n^*$ such that r is equivariant with respect to the diagonal action of $\mathrm{Aut}_f(X)$ on the product of the trees Λ_i^*;*

(2) *an automorphism $\phi \in \mathrm{Aut}_f(X)$ is elliptic if and only if the action of ϕ on each of the trees Λ_i^* is elliptic and axial if and only if the action of ϕ on at least one of the trees Λ_i^* is axial;*

(3) *if $\Gamma \subset \mathrm{Aut}_f(X)$ is a subgroup that does not have a fixed point in X, then Γ acts without fixed point on at least one of the trees Λ_i^*.*

The next result is a version of the Tits Alternative on the existence of free subgroups. Our result extends and our proof relies on a corresponding result for the action of a group Γ on a tree T: if Γ does not fix a point or an end or a pair of ends of T, then Γ contains a free nonabelian subgroup acting freely on T, see [PV].

THEOREM 2. *Let X be an n-dimensional simply connected foldable cubical chamber complex of nonpositive curvature and $\Gamma \subset \mathrm{Aut}(X)$ a subgroup. Suppose that Γ does not contain a free nonabelian subgroup acting freely on X. Then up to passing to a subgroup of finite index, there is a surjective homomorphism $h\colon \Gamma \to \mathbf{Z}^k$ for some $k \in \{0, \dots, n\}$ such that the kernel Δ of h consists precisely of the elliptic elements of Γ and, furthermore, precisely one of the following three possibilities occurs:*

(1) *Γ fixes a point in X (then $k = 0$).*

(2) *$k \geq 1$ and there is a Γ-invariant convex subset $E \subset X$ isometric to k-dimensional Euclidean space such that Δ fixes E pointwise and such that Γ/Δ acts on E as a cocompact lattice of translations. In particular, Γ fixes each point of $E(\infty) \subset X(\infty)$.*

(3) *Γ fixes a point of $X(\infty)$, but Δ does not fix a point in X. There is a sequence (x_m) in X with strictly increasing stabilizers, $\mathrm{Stab}_{x_m} \subsetneq \mathrm{Stab}_{x_{m+1}}$, with $\bigcup \mathrm{Stab}_{x_m} = \Delta$. Up to passing to a subsequence, any such sequence converges to a fixed point of Γ in $X(\infty)$.*

If the action of Γ is free or, more generally, if there is a universal upper bound on the order of the stabilizers of the action, then possibility (3) in Theorem 2 cannot occur.

Related to Property (T) there is the question of the existence of an epimorphism onto the group $\mathbf{Z}$ of integers for a finite index subgroup of a group. Recently, C. Gonciulea gave a positive answer to this question in the case of infinite Coxeter groups [Go]. We give a positive answer for groups acting on a class of cubical manifolds.

THEOREM 3. *Let X be a simply connected cubical manifold of nonpositive curvature and assume that the number of chambers adjacent to each face of codimension 2 in X is divisible by 4. Let Γ be a group acting on X cocompactly by automorphisms. Then a finite index subgroup of Γ admits a surjective homomorphism onto $\mathbf{Z}^n$, where $n = \dim X$.*

The paper is organized as follows. In Section 1 we recall basic definitions and facts related to cubical complexes, prove some criteria for foldability and discuss nonpositive curvature. In Section 2 we recall some constructions and examples of foldable cubical complexes. In Section 3, we introduce hyperplanes in cubical complexes as in [NR]. Foldability then leads to systems of disjoint hyperplanes and their "dual trees" which will accomplish the proof of Theorem 1. In Section 4 we investigate the induced actions on the dual trees and obtain the proof of Theorem 2. In Section 5 we develop the idea of parallel transport in cubical manifolds and use it to prove Theorem 3.

We are grateful to M. Bridson, T. Januszkiewicz, S. Mozes and the referee for helpful discussions and hints.

1. Cubical complexes

In this section we briefly recall basic notions and facts related to cubical complexes.

Cubical complexes and cubical metric

A *cell* P is the convex hull of a finite set of points in a real vector space. Faces of P are then well defined, and they are also cells (see e.g. [Br]). The set $\mathcal{P}$ of faces of P is partially ordered by inclusion and called the *poset* of P. Two cells are *combinatorially equivalent* if their posets are isomorphic. For example, every convex quadrilateral polygon is combinatorially equivalent to the unit square. An isomorphism of posets induces a bijection between sets of barycenters of faces and thus determines a piecewise linear homeomorphism between two cells. We call such a homeomorphism a *realization* of a combinatorial equivalence.

A *cell complex* is a collection X of cells which are glued by realizations of combinatorial equivalences along faces. We also assume that different faces of the same cell are not identified and that the intersection of different cells is either empty or consists of one cell. These latter assumptions are not essential, but they simplify the exposition considerably. However, we do not require that X is locally finite, so that, if not explicitely stated otherwise, a vertex in X may belong to infinitely many distinct cells.

We say that a cell complex X is *simplicial* if the cells of X are simplices. Because of our assumptions on the glueing of faces, this coincides with the standard terminology. We say that X is *cubical* if the cells of X are combinatorially equivalent to cubes.