

7.2 Cycles

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **45 (1999)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **25.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Solving the system, we obtain

$$F(u, t) = \frac{1 + (1 - u)t}{1 - (v - 2 + u)t} \cdot \frac{1 - (v - 2)t + (1 - u)(v - 2 + u)t^2}{1 + t + (1 - u)(v - 2 + u)t^2}.$$

We then compute

$$\begin{aligned} G(t) &= F(1, t) = \frac{1 - (v - 2)t}{(1 + t)(1 - (v - 1)t)}, \\ F(0, t) &= \frac{(1 + t)(1 - (v - 2)t + (v - 2)t^2)}{(1 - (v - 2)t)(1 + t + (v - 2)t^2)}. \end{aligned}$$

7.2 CYCLES

Let $\mathcal{X} = C_k$, the cycle on k vertices. Here, as there are 2 proper circuits of length n for all n multiple of k (except 0), we have

$$F(0, t) = \frac{1 + t^k}{1 - t^k}.$$

Obtaining a closed form for G is much harder. The number of circuits of length n is

$$g_n = \sum_{i \in \mathbf{Z} : i \equiv 0 \pmod{k}, i \equiv n \pmod{2}} \binom{n}{\frac{n+i}{2}},$$

from which, by [Gou72, 1.54], it follows that

$$G(t) = \frac{1}{k} \sum_{\zeta^k=1} \frac{1}{1 - (\zeta + \zeta^{-1})t} = \frac{1}{k} \sum_{j=0}^{k-1} \frac{1}{1 - 2 \cos\left(\frac{2\pi j}{k}\right)t}.$$

It is not at all obvious how to simplify the above expression. A closed-form answer can be obtained from (2.3), namely

$$G(t) = \frac{(2t)^2 + (1 - \sqrt{1 - 4t^2})^2}{(2t)^2 - (1 - \sqrt{1 - 4t^2})^2} \cdot \frac{(2t)^k + (1 - \sqrt{1 - 4t^2})^k}{(2t)^k - (1 - \sqrt{1 - 4t^2})^k},$$

or, expanding,

$$G(t) = \frac{(2t)^k + \sum_{m=0}^{k/2} (1 - 4t^2)^m \binom{k}{2m}}{\sum_{m=1}^{(k+1)/2} (1 - 4t^2)^m \binom{k}{2m-1}}.$$

However in general this fraction is not reduced. To obtain reduced fractions for $F(u, t)$ (and thus for $G(t)$), we have to consider separately the cases where k is odd or even.

For odd k , letting $k = 2\ell + 1$, we obtain

$$F(u, t) = \frac{1 + (1 - u)t}{1 - (1 + u)t} \cdot \frac{\sum_{m=0}^{\ell} \alpha_m^{\ell} (-t)^m (1 + (1 - u^2)t^2)^{\ell-m}}{\sum_{m=0}^{\ell} \alpha_m^{\ell} t^m (1 + (1 - u^2)t^2)^{\ell-m}},$$

$$G(t) = \frac{\sum_{m=0}^{\ell} \alpha_m^{\ell} (-t)^m}{(1 - 2t) \left(\sum_{m=0}^{\ell} \alpha_m^{\ell} t^m \right)},$$

where

$$\alpha_m^{\ell} = \begin{cases} (-)^{\frac{m}{2}} \binom{\ell - \frac{m}{2}}{\frac{m}{2}} & \text{if } m \equiv 0 [2], \\ (-)^{\frac{m-1}{2}} \binom{\ell - \frac{m+1}{2}}{\frac{m-1}{2}} & \text{if } m \equiv 1 [2]. \end{cases}$$

For even k , with $k = 2\ell$,

$$F(u, t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell-m} \binom{\ell-m}{m} (-t^2)^m (1 - (1 - u^2)t^2)^{\ell-2m}}{(1 - (1 + u)^2 t^2) \left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell-1-m}{m} (-t^2)^m (1 - (1 - u^2)t^2)^{\ell-1-2m} \right)},$$

$$G(t) = \frac{\sum_{m=0}^{\ell/2} \frac{\ell}{\ell-m} \binom{\ell-m}{m} (-t^2)^m}{(1 - 4t^2) \left(\sum_{m=0}^{(\ell-1)/2} \binom{\ell-1-m}{m} (-t^2)^m \right)},$$

expressed as reduced fractions.

The first few values of F , where \square stands for $1 + (1 - u^2)t^2$, are:

k	$F(u, t)$	k	$F(u, t)$
1	$\frac{1 + (1 - u)t}{1 - (1 + u)t}$	2	$\frac{\square}{1 - (1 + u)^2 t^2}$
3	$\frac{(1 + (1 - u)t)(\square - t)}{(1 - (1 + u)t)(\square + t)}$	4	$\frac{\square^2 - 2t^2}{1 - (1 + u)^2 t^2}$
5	$\frac{(1 + (1 - u)t)(\square^2 - \square t - t^2)}{(1 - (1 + u)t)(\square^2 + \square t + t^2)}$	6	$\frac{\square^2 - 3t^2}{(1 - (1 + u)^2 t^2)(\square^2 - t^2)}$
7	$\frac{(1 + (1 - u)t)(\square^3 - \square^2 t - 2\square t^2 + t^3)}{(1 - (1 + u)t)(\square^3 + \square^2 t - 2\square t^2 - t^3)}$	8	$\frac{\square^4 - 4\square^2 t^2 + 2t^4}{(1 - (1 + u)^2 t^2)(\square^2 - 2t^2)}$
9	$\frac{(1 + (1 - u)t)(\square - t)(\square^3 - 3\square t^2 - t^3)}{(1 - (1 + u)t)(\square + t)(\square^3 - 3\square t^2 + t^3)}$	10	$\frac{\square^4 - 5\square^2 t^2 + 5t^4}{(1 - (1 + u)^2 t^2)(\square^4 - 3\square^2 t^2 + t^4)}$

These rational expressions were computed and simplified using the computer algebra program *Maple*TM.

7.3 TREES

Let \mathcal{X} be the d -regular tree. Then

$$F(0, t) = 1$$

as a tree has no proper circuit; while direct (i.e., without using Corollary 2.6) computation of G is more complicated. It was first performed by Kesten [Kes59]; here we will derive the extended circuit series $F(u, t)$ and also obtain the answer using Corollary 2.6.

Let \mathcal{T} be a regular tree of degree d with a fixed root \star , and let \mathcal{T}' be the connected component of \star in the two-tree forest obtained by deleting in \mathcal{T} an edge at \star . Let $F(u, t)$ and $F'(u, t)$ respectively count circuits at \star in \mathcal{T} and \mathcal{T}' . For instance if $d = 2$ then F' counts circuits in \mathbf{N} and F counts circuits in \mathbf{Z} . For a reason that will become clear below, we make the convention that the empty circuit is counted as ‘1’ in F and as ‘ u ’ in F' . Then we have

$$F' = u + (d - 1)tF't \frac{1}{1 - (d - 2 + u)tF't},$$

$$F = 1 + dtF't \frac{1}{1 - (d - 1 + u)tF't}.$$

Indeed a circuit in \mathcal{T}' is either the empty circuit (counted as u), or a sequence of circuits composed of, first, a step in any of $d - 1$ directions, then