2. Preliminaries

THEOREM 4.12. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \le |pq^{-1}F_0|_p$. Then the quantity*

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.

These results show that if related congruences hold for

$$\beta_{n,\chi}(0) = -\frac{1}{n}(1 - \chi_n(p)p^{n-1})B_{n,\chi_n},$$

then they must also hold for $\beta_{n,\chi}(\tau)$, where τ is any element of $\mathbf{Z}_p$ such that $|\tau|_p \le |pq^{-1}F_0|_p$.

In [9] Granville defined ordinary Bernoulli numbers of negative index, B_{-n}, where $n \in \mathbf{Z}$, $n \ge 1$, in the field $\mathbf{Q}_p$ according to

$$B_{-n} = \lim_{k\to\infty} B_{\phi(p^k)-n},$$

where the limit is taken in the p-adic sense. In a similar manner we define generalized Bernoulli numbers of negative index, $B_{-n,\chi}$, $n \in \mathbf{Z}$, $n \ge 1$, and a collection of functions that correspond to generalized Bernoulli polynomials of negative index, $B_{-n,\chi}(t)$, $n \in \mathbf{Z}$, $n \ge 1$. As a result of our definitions, we show that the $B_{-n,\chi}(t)$ are actually power series that can be written in the form

$$B_{-n,\chi}(t) = \sum_{m=0}^{\infty}\binom{-n}{m}B_{-n-m,\chi}t^m,$$

converging for $t \in \mathbf{C}_p$, $|t|_p < 1$. We close out by considering some properties of these functions.

2. PRELIMINARIES

The p-adic L-functions, $L_p(s;\chi)$, were first generated by Kubota and Leopoldt for the purpose of finding functions that would serve as analogues of the Dirichlet L-functions in the p-adic number field [14]. They are characterized by the fact that they interpolate a specific expression involving generalized Bernoulli numbers when the variable s is a nonpositive integer. In the following, for each $\tau \in \mathbf{C}_p$, $|\tau|_p \le 1$, we derive a p-adic function $L_p(s,\tau;\chi)$ that interpolates a specific expression involving generalized

Bernoulli polynomials in τ for similar values of the variable s. These functions are designed so that $L_p(s,0;\chi) = L_p(s;\chi)$. The method of derivation follows that found in [13], Chapter 3. However, this method will only account for those $\tau \in \overline{\mathbf{Q}}_p$ with $|\tau|_p \leq 1$. To complete the derivation we show that there exist functions $L_p(s,\tau;\chi)$ for all $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, such that for every sequence $\{\tau_i\}_{i=0}^{\infty}$ in $\overline{\mathbf{Q}}_p$, with $|\tau_i|_p \leq 1$, converging to some $\tau \in \mathbf{C}_p$, the sequence $\{L_p(1-n,\tau_i;\chi)\}_{i=0}^{\infty}$, with $n \in \mathbf{Z}$, $n \geq 1$, converges to $L_p(1-n,\tau;\chi)$. Thus for each $\tau \in \mathbf{C}_p$, $|\tau|_p \leq 1$, the function $L_p(s,\tau;\chi)$ must interpolate the appropriate expressions involving generalized Bernoulli polynomials for $s = 1-n$, $n \in \mathbf{Z}$, $n \geq 1$.

Before we begin the derivation, we must first define the concepts that we shall need and review some of their resulting properties.

2.1 Dirichlet characters

For $n \in \mathbf{Z}$, $n \geq 1$, a Dirichlet character to the modulus n is a multiplicative map $\chi : \mathbf{Z} \to \mathbf{C}$ such that $\chi(a+n) = \chi(a)$ for all $a \in \mathbf{Z}$, and $\chi(a) = 0$ if and only if $(a,n) \neq 1$. Since $a^{\phi(n)} \equiv 1 \pmod{n}$ for all a such that $(a,n) = 1$, $\chi(a)$ must be a root of unity for such a.

If χ is a Dirichlet character to the modulus n, then for any positive multiple m of n we can induce a Dirichlet character ψ to the modulus m according to

$$\psi(a) = \begin{cases} \chi(a), & \text{if } (a,m) = 1 \\ 0, & \text{if } (a,m) \neq 1. \end{cases}$$

The minimum modulus n for which a character χ cannot be induced from some character to the modulus m, $m < n$, is called the conductor of χ, denoted f_χ. We shall assume that each χ is defined modulo its conductor. Such a character is said to be primitive.

For primitive Dirichlet characters χ and ψ having conductors f_χ and f_ψ, respectively, we define the product, $\chi\psi$, to be the primitive character with $\chi\psi(a) = \chi(a)\psi(a)$ for all $a \in \mathbf{Z}$ such that $(a, f_\chi f_\psi) = 1$. Note that there may exist some values of a such that $\chi\psi(a) \neq \chi(a)\psi(a)$, due to the fact that our definition requires $\chi\psi$ to be a primitive character. The conductor $f_{\chi\psi}$ then divides $\operatorname{lcm}(f_\chi, f_\psi)$. With this operation defined, we can then consider the set of primitive Dirichlet characters to form a group under multiplication. The identity of the group is the principal character $\chi = 1$, having conductor $f_1 = 1$. The inverse of the character χ is the character $\chi^{-1} = \overline{\chi}$, the map of complex conjugates of the values of χ.

Since any Dirichlet character χ is multiplicative, we must have $\chi(-1) = \pm 1$. A character χ is said to be odd if $\chi(-1) = -1$, and even if $\chi(-1) = 1$.

2.2 Generalized Bernoulli polynomials

Let χ be a Dirichlet character with conductor f_χ. Then we define the functions, $B_{n,\chi}(t)$, $n \in \mathbf{Z}$, $n \geq 0$, by the generating function

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}(t)\frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi}. \tag{1}$$

We define the generalized Bernoulli numbers associated with χ, $B_{n,\chi}$, $n \in \mathbf{Z}$, $n \geq 0$, by

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{x^n}{n!}, \quad |x| < \frac{2\pi}{f_\chi},$$

so that $B_{n,\chi}(0) = B_{n,\chi}$. Note that

$$\sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{(a+t)x}}{e^{f_\chi x} - 1} = e^{tx} \sum_{a=1}^{f_\chi} \frac{\chi(a)xe^{ax}}{e^{f_\chi x} - 1},$$

which implies that

$$\sum_{n=0}^{\infty} B_{n,\chi}(t)\frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} B_{n,\chi}\frac{x^n}{n!},$$

and from this we obtain

$$B_{n,\chi}(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m,\chi}t^m. \tag{2}$$

Thus the functions $B_{n,\chi}(t)$, defined in (1), are actually polynomials, called the generalized Bernoulli polynomials associated with χ. Let $\mathbf{Z}[\chi]$ denote the ring generated over $\mathbf{Z}$ by all the values $\chi(a)$, $a \in \mathbf{Z}$, and $\mathbf{Q}(\chi)$ the field generated over $\mathbf{Q}$ by all such values. Then it can be shown that $f_\chi B_{n,\chi}$ must be in $\mathbf{Z}[\chi]$ for each $n \geq 0$ whenever $\chi \neq 1$. In general, we have $B_{n,\chi} \in \mathbf{Q}(\chi)$ for each $n \geq 0$, and so $B_{n,\chi}(t) \in \mathbf{Q}(\chi)[t]$. The polynomials $B_{n,\chi}(t)$ exhibit the property that, for all $n \geq 0$,

$$B_{n,\chi}(-t) = (-1)^n\chi(-1)B_{n,\chi}(t), \tag{3}$$

whenever $\chi \neq 1$. Thus $B_{n,\chi}(t)$, for $\chi \neq 1$, is either an even function or an odd function according to whether $(-1)^n\chi(-1)$ is 1 or -1. From (3) we obtain

$$B_{n,\chi} = (-1)^n \chi(-1) B_{n,\chi}\,,$$

and so $B_{n,\chi} = 0$ whenever n is even and χ is odd, or whenever n is odd and χ is even, $\chi \neq 1$. Another property that the polynomials satisfy is that for $m \in \mathbf{Z}$, $m \geq 1$,

$$B_{n,\chi}(mf_\chi + t) - B_{n,\chi}(t) = n \sum_{a=1}^{mf_\chi} \chi(a)(a+t)^{n-1}, \tag{4}$$

for all $n \geq 0$. This can be derived from (1). Note that for $\chi = 1$ and $t = 0$ this becomes

$$\frac{1}{n}\left(B_{n,1}(m) - B_{n,1}\right) = \sum_{a=1}^{m} a^{n-1}.$$

If $\chi \neq 1$, then it can be shown that $\sum_{a=1}^{f_\chi} \chi(a) = 0$, and from the above relations we can derive

$$B_{0,\chi} = \frac{1}{f_\chi} \sum_{a=1}^{f_\chi} \chi(a)$$

for all χ. Therefore

$$B_{0,\chi} = \begin{cases} 0, & \text{if } \chi \neq 1 \\ 1, & \text{if } \chi = 1\,. \end{cases}$$

The ordinary Bernoulli polynomials, $B_n(t)$, $n \in \mathbf{Z}$, $n \geq 0$, are defined by

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!}, \qquad |x| < 2\pi, \tag{5}$$

and the Bernoulli numbers, B_n, $n \in \mathbf{Z}$, $n \geq 0$,

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \qquad |x| < 2\pi\,.$$

From this we obtain the values $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30, \ldots$, with $B_n = 0$ for odd $n \geq 3$. For even $n \geq 2$, we have

$$B_n = -\frac{1}{n+1} \sum_{m=0}^{n-1} \binom{n+1}{m} B_m\,.$$

Note that we again have the relations $B_n(0) = B_n$ and

$$B_n(t) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m} t^m,$$

as we did for the generalized Bernoulli polynomials.

Some of the more important properties of Bernoulli polynomials are that

$$B_n(t+1) - B_n(t) = nt^{n-1} \tag{6}$$

for all $n \geq 1$, and

$$B_n(1-t) = (-1)^n B_n(t)$$

for $n \geq 0$. Each of these results can be derived from the generating function (5) above.

Similar to (4) for the generalized Bernoulli polynomials, whenever $m, n \in \mathbf{Z}$, $m \geq 1$, $n \geq 1$,

$$\frac{1}{n}(B_n(m) - B_n) = \sum_{a=0}^{m-1} a^{n-1},$$

where we take 0^0 to be 1 in the case of $a = 0$ and $n = 1$. Note that this can be derived from (6) since

$$B_n(m) - B_n = \sum_{a=0}^{m-1} (B_n(a+1) - B_n(a)) .$$

The Bernoulli numbers are rational numbers, and, in fact, the von Staudt-Clausen theorem states that for even $n \geq 2$,

$$B_n + \sum_{\substack{p \text{ prime} \\ (p-1)|n}} \frac{1}{p} \in \mathbf{Z}.$$

Thus the denominator of each B_n must be square-free.

The ordinary Bernoulli numbers are related to the generalized Bernoulli numbers in that for $\chi = 1$ we have

$$\frac{xe^x}{e^x - 1} = \sum_{n=0}^{\infty} B_{n,1} \frac{x^n}{n!}, \quad |x| < 2\pi,$$

and since

$$\frac{xe^x}{e^x - 1} = x + \frac{x}{e^x - 1},$$

we see that $B_{n,1} = B_n$ for all $n \neq 1$, and $B_{1,1} = -B_1$. In fact, this can be written as $B_{n,1} = (-1)^n B_n$, and for the polynomials, $B_{n,1}(t) = (-1)^n B_n(-t)$.

2.3 Dirichlet L-functions

For χ a Dirichlet character with conductor f_χ, the Dirichlet L-function for χ is defined by

$$L(s;\chi) = \sum_{b=1}^{\infty} \frac{\chi(b)}{b^s},$$

for $s \in \mathbf{C}$ such that $\Re(s) > 1$. Note that $L(s;\chi)$ can be continued analytically to all of $\mathbf{C}$, except for a pole of order 1 at $s = 1$ when $\chi = 1$.

Let $\tau(\chi)$ be a Gauss sum,

$$\tau(\chi) = \sum_{a=1}^{f_\chi} \chi(a) e^{2\pi i a/f_\chi},$$

where $i^2 = -1$, and let

$$\delta_\chi = \begin{cases} 0, & \text{if } \chi(-1) = 1 \\ 1, & \text{if } \chi(-1) = -1 . \end{cases}$$

Then $L(s;\chi)$ satisfies the functional equation

$$(7) \quad \left(\frac{f_\chi}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\delta_\chi}{2}\right) L(s;\chi) = W_\chi \left(\frac{f_\chi}{\pi}\right)^{(1-s)/2} \Gamma\left(\frac{1-s+\delta_\chi}{2}\right) L(1-s;\overline{\chi}),$$

where $\Gamma(s)$ is the gamma function, and $W_\chi = \frac{\tau(\chi)}{i^{\delta_\chi} \cdot \sqrt{f_\chi}}$, having the property that $|W_\chi| = 1$. Since $\Gamma(s)$ has simple poles at the negative integers, $L(s;\chi)$ must be zero for $s = 1 - n$, where $n \in \mathbf{Z}$, $n \geq 1$, such that $n \not\equiv \delta_\chi \pmod 2$, except when $\chi = 1$ and $n = 1$. $L(s;\chi)$ can also be described by means of the Euler product $L(s;\chi) = \prod_{p \text{ prime}} \left(1 - \chi(p)p^{-s}\right)^{-1}$, for $s \in \mathbf{C}$ such that $\Re(s) > 1$. Thus $L(s;\chi) \neq 0$ in this domain.

The generalized Bernoulli numbers, $B_{n,\chi}$, and the Dirichlet L-function, $L(s;\chi)$, share the following relationship, a proof of this being found in [13]:

THEOREM 2.1. *Let χ be a Dirichlet character, and let $n \in \mathbf{Z}$, $n \geq 1$. Then $L(1-n;\chi) = -\frac{1}{n} B_{n,\chi}$.*

Thus we have a way to express certain values of a function defined in terms of an infinite sum as quantities that can be found by a finite process.

2.4 THE p-ADIC NUMBER FIELD

Let p be prime. We shall use $\mathbf{Z}_p$ to represent the p-adic integers, and $\mathbf{Q}_p$ the p-adic rationals. Let $|\cdot|_p$ denote the p-adic absolute value on $\mathbf{Q}_p$, normalized so that $|p|_p = p^{-1}$. Let $\overline{\mathbf{Q}}_p$ be the algebraic closure of $\mathbf{Q}_p$. The absolute value on $\mathbf{Q}_p$ extends uniquely to $\overline{\mathbf{Q}}_p$, however $\overline{\mathbf{Q}}_p$ is not complete with respect to the absolute value. Let $\mathbf{C}_p$ be the completion of $\overline{\mathbf{Q}}_p$ with respect to this absolute value. Then the absolute value extends to $\mathbf{C}_p$, and $\overline{\mathbf{Q}}_p$ is dense in $\mathbf{C}_p$. We also have $\mathbf{C}_p$ algebraically closed. Furthermore, on $\mathbf{C}_p$ the absolute value is non-Archimedean, and so

$$|a+b|_p \leq \max\{|a|_p, |b|_p\}$$

for any $a, b \in \mathbf{C}_p$. Note that the two fields $\mathbf{C}$ and $\mathbf{C}_p$ are algebraically isomorphic, and any one of the two can be embedded in the other. We denote two particular subrings of $\mathbf{C}_p$ in the following manner

$$\mathfrak{o} = \{a \in \mathbf{C}_p : |a|_p \leq 1\}, \qquad \mathfrak{p} = \{a \in \mathbf{C}_p : |a|_p < 1\}.$$

Then $\mathfrak{p}$ is a maximal ideal of $\mathfrak{o}$. If $\tau \in \mathbf{C}_p$ such that $|\tau|_p \leq |p|_p^s$, where $s \in \mathbf{Q}$, then $\tau \in p^s\mathfrak{o}$, and so we shall also write this as $\tau \equiv 0 \pmod{p^s\mathfrak{o}}$.

Any $n \in \mathbf{Z}$, $n > 0$, can be uniquely expressed in the form $n = \sum_{m=0}^{k} a_m p^m$, where $a_m \in \mathbf{Z}$, $0 \leq a_m \leq p-1$, for $m = 0, 1, \ldots, k$, and $a_k \neq 0$. For such n, we define

$$s_p(n) = \sum_{m=0}^{k} a_m,$$

the sum of the p-adic digits of n, and also define $s_p(0) = 0$. For any $n \in \mathbf{Z}$, let $v_p(n)$ be the highest power of p dividing n. This function is additive, and relates to the function $s_p(n)$ by means of the identity

$$v_p(n!) = \frac{n - s_p(n)}{p-1}, \tag{8}$$

which holds for all $n \geq 0$. Note that for $n \geq 1$ this implies that

$$v_p(n!) \leq \frac{n-1}{p-1}.$$

The definition of this function can be extended to all of $\mathbf{Q}$ by taking $v_p(1/n) = -v_p(n)$.

Throughout we let $q = 4$ if $p = 2$, and $q = p$ otherwise. Note that there exist $\phi(q)$ distinct solutions, modulo q, to the equation $x^{\phi(q)} - 1 = 0$, and each solution must be congruent to one of the values $a \in \mathbf{Z}$, where $1 \leq a \leq q$,

$(a,p) = 1$. Thus, by Hensel's Lemma, given $a \in \mathbf{Z}$ with $(a,p) = 1$, there exists a unique $\omega(a) \in \mathbf{Z}_p$, where $\omega(a)^{\phi(q)} = 1$, such that

$$\omega(a) \equiv a \pmod{q\mathbf{Z}_p}.$$

Letting $\omega(a) = 0$ for $a \in \mathbf{Z}$ such that $(a,p) \neq 1$, we see that ω is actually a Dirichlet character, called the Teichmüller character, having conductor $f_\omega = q$. Let us define

$$\langle a \rangle = \omega^{-1}(a)a.$$

Then $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$. For $p \geq 3$, $\lim_{n\to\infty} a^{p^n} = \omega(a)$, since $a^{p^n} \equiv a \pmod{p}$ and $a^{p^n(p-1)} \equiv 1 \pmod{p^{n+1}}$.

For our purposes we shall need to make a slight extension of the definition of the Teichmüller character ω. If $t \in \mathbf{C}_p$ such that $|t|_p \leq 1$, then for any $a \in \mathbf{Z}$, $a + qt \equiv a \pmod{q\mathfrak{o}}$. Thus we define

$$\omega(a + qt) = \omega(a)$$

for these values of t. We also define

$$\langle a + qt \rangle = \omega^{-1}(a)(a + qt)$$

for such t.

Fix an embedding of the algebraic closure of $\mathbf{Q}$, $\overline{\mathbf{Q}}$, into $\mathbf{C}_p$. We may then consider the values of a Dirichlet character χ as lying in $\mathbf{C}_p$. For $n \in \mathbf{Z}$ we define the product $\chi_n = \chi\omega^{-n}$ in the sense of the product of characters. This implies that $f_{\chi_n} \mid f_\chi q$. However, since we can write $\chi = \chi_n \omega^n$, we also have $f_\chi \mid f_{\chi_n} q$. Thus f_χ and f_{χ_n} differ by a factor that is a power of p. In fact, either $f_{\chi_n}/f_\chi \in \mathbf{Z}$ and divides q, or $f_\chi/f_{\chi_n} \in \mathbf{Z}$ and divides q.

Let $\mathbf{Q}_p(\chi)$ denote the field generated over $\mathbf{Q}_p$ by all values $\chi(a)$, $a \in \mathbf{Z}$. In this context we can state the following, found in [13] (pp. 14–15).

LEMMA 2.2. *In the field $\mathbf{Q}_p(\chi)$, for all $n \in \mathbf{Z}$, $n \geq 0$,*

$$B_{n,\chi} = \frac{1}{n+1} \lim_{h\to\infty} \frac{1}{p^h f_\chi} \left(B_{n+1,\chi}\left(p^h f_\chi\right) - B_{n+1,\chi}(0) \right).$$

From this we can obtain

LEMMA 2.3. *Let $\tau \in \mathbf{C}_p$. In the field $\mathbf{Q}_p(\chi, \tau)$, for all $n \in \mathbf{Z}$, $n \geq 0$,*

$$B_{n,\chi_n}(\tau) = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a+\tau)^n .$$

Proof. By applying Lemma 2.2 to (4), we obtain

$$B_{n,\chi} = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi(a) a^n .$$

Therefore, by (2),

$$\begin{aligned} B_{n,\chi_n}(\tau) &= \sum_{m=0}^{n} \binom{n}{m} \tau^{n-m} \lim_{h\to\infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a) a^m \\ &= \lim_{h\to\infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a) \sum_{m=0}^{n} \binom{n}{m} \tau^{n-m} a^m . \end{aligned}$$

Since f_χ and f_{χ_n} differ by a factor that is a power of p, we must have

$$B_{n,\chi_n}(\tau) = \lim_{h\to\infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a+\tau)^n,$$

and the proof is complete. □

2.5 p-ADIC FUNCTIONS

Let K be an extension of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$. An infinite series $\sum_{n=0}^{\infty} a_n$, $a_n \in K$, converges in K if and only if $|a_n|_p \to 0$ as $n \to \infty$. Let $K[[x]]$ be the algebra of formal power series in x. Then it follows that a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

in $K[[x]]$, converges at $x = \xi$, $\xi \in \mathbf{C}_p$, if and only if $|a_n \xi^n|_p \to 0$ as $n \to \infty$. Therefore whenever a power series $A(x)$ converges at some $\xi_0 \in \mathbf{C}_p$, then it must converge at all $\xi \in \mathbf{C}_p$ such that $|\xi|_p \leq |\xi_0|_p$. The following result, for double series in K, can be found in [8].

PROPOSITION 2.4. *Let $b_{n,m} \in K$, and suppose that for each $\epsilon > 0$ there exists $N \in \mathbf{Z}$, depending on ϵ, such that if* $\max\{n, m\} \geq N$, *then $|b_{n,m}|_p \leq \epsilon$. Then both series*

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} b_{n,m}\right) \quad \textit{and} \quad \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} b_{n,m}\right)$$

converge, and their sums are equal.

There are two power series that we wish to make note of in particular. First we define the p-adic exponential function, $\exp(x)$, in $\mathbf{Q}_p[[x]]$, by

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n. \tag{9}$$

From (8) we can conclude that this power series converges in $\{x \in \mathbf{C}_p : |x|_p < p^{-1/(p-1)}\}$. The p-adic logarithm function, $\log(x)$, in $\mathbf{Q}_p[[x]]$, is defined by

$$\log(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \tag{10}$$

the power series converging in the domain $\{x \in \mathbf{C}_p : |x|_p < 1\}$. For $|x|_p < p^{-1/(p-1)}$, we have $\log(\exp(x)) = x$ and $\exp(\log(1 + x)) = 1 + x$.

The following property is a uniqueness property for power series, found in [13].

LEMMA 2.5. *Let $A(x), B(x) \in K[[x]]$, such that each converges in a neighborhood of 0 in $\mathbf{C}_p$. If $A(\xi_n) = B(\xi_n)$ for a sequence $\{\xi_n\}_{n=0}^{\infty}$, $\xi_n \neq 0$, in $\mathbf{C}_p$, such that $\xi_n \to 0$, then $A(x) = B(x)$.*

Let U be an open subset of $\mathbf{C}_p$, contained in the domain of the p-adic function f. We say that f is differentiable at $x \in U$ if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

exists. If this limit exists for each $x \in U$, then we say that f is differentiable in U.

The relationship between the derivatives of a function and its power series expansion is given in the following result, found in [8].

PROPOSITION 2.6. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with coefficients in $\mathbf{C}_p$, and suppose that*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-\alpha)^n$$

converges on some closed ball B in $\mathbf{C}_p$. Then

i) *For each $x \in B$, the k^{th} derivative $f^{(k)}(x)$ exists, and is given by*

$$f^{(k)}(x) = k! \sum_{n=k}^{\infty} \binom{n}{k} a_n (x-\alpha)^{n-k},$$

and we have

$$a_k = \frac{1}{k!} f^{(k)}(\alpha).$$

ii) *Let $\beta \in B$. Then there exists a series $\sum_{n=0}^{\infty} b_n x^n$ such that*

$$f(x) = \sum_{n=0}^{\infty} b_n (x-\beta)^n$$

for any $x \in B$. Both series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have the same region of convergence.

Now let K be a finite extension of $\mathbf{Q}_p$. For $A(x) \in K[[x]]$, $A(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_n \in K$, define

$$\|A\| = \sup_n |a_n|_p .$$

Let $P_K = \{A(x) \in K[[x]] : \|A\| < \infty\}$. Then $\|\cdot\|$ defines a norm on P_K, and so $K[x] \subset P_K \subset K[[x]]$. Furthermore P_K is complete in this norm.

Let $\{b_n\}_{n=0}^{\infty}$ be a sequence of elements of K, and let the sequence $\{c_n\}_{n=0}^{\infty}$ be defined by

$$c_n = \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} b_m \tag{11}$$

for each $n \in \mathbf{Z}$, $n \geq 0$. Then $c_n \in K$ for each $n \geq 0$. Note that (11) implies that these sequences must satisfy

$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = e^{-t} \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} .$$

This implies that

$$\sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = e^t \sum_{n=0}^{\infty} c_n \frac{t^n}{n!},$$

and so we have the relationship

$$b_n = \sum_{m=0}^{n} \binom{n}{m} c_m \tag{12}$$

for each $n \in \mathbf{Z}$, $n \geq 0$. We can reverse this process to derive (11) given (12). Thus (11) and (12) must be equivalent. The following relate to sequences that satisfy (11) and (12), and are found in [13].

THEOREM 2.7. *Let $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ be defined as in the above relation. Let $\rho \in \mathbf{R}$ such that $0 < \rho < |p|_p^{1/(p-1)}$. If $|c_n|_p \leq C\rho^n$ for all $n \geq 0$, where $C > 0$, then there exists a unique power series $A(x) \in P_K$ such that $A(x)$ converges at every $\xi \in \mathbf{C}_p$ with $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$, and $A(n) = b_n$ for every $n \geq 0$.*

COROLLARY 2.8. *Let $A(x)$ be the power series from the theorem. Then for each $\xi \in \mathbf{C}_p$ such that $|\xi|_p < |p|_p^{1/(p-1)}\rho^{-1}$, we have*

$$A(\xi) = \sum_{n=0}^{\infty} c_n \binom{\xi}{n}.$$

Theorem 2.7 can be applied to the sequence $\{b_n\}_{n=0}^{\infty}$ in $K = \mathbf{Q}_p(\chi)$, where

$$b_n = \left(1 - \chi_n(p)p^{n-1}\right) B_{n,\chi_n},$$

in order to obtain a power series $A_\chi(s)$ satisfying $A_\chi(n) = b_n$, and converging on the domain $\{s \in \mathbf{C}_p : |s|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$. (Since $|p|_p^{1/(p-1)}|q|_p^{-1} > 1$ and $|n|_p \leq 1$ for each $n \in \mathbf{Z}$, all of $\mathbf{Z}$ is contained in this domain.) From this a p-adic function, $L_p(s;\chi)$, can be derived that interpolates the values

$$L_p(1-n;\chi) = -\frac{1}{n} b_n,$$

and which converges in $\{s \in \mathbf{C}_p : |s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}\}$, except $s \neq 1$ if $\chi = 1$. Note that if χ is odd, then χ_n is even when n is odd, and χ_n is odd when n is even. Thus the quantity $(1-\chi_n(p)p^{n-1})B_{n,\chi_n} = 0$ for all $n \in \mathbf{Z}$, $n \geq 1$, as we saw from the properties of generalized Bernoulli numbers. Therefore $L_p(s;\chi)$ vanishes on a sequence such as $\{-p^m\}_{m=0}^{\infty}$, which has 0 as a limit point, implying that for such χ we must have $L_p(s;\chi) \equiv 0$.