

## 2.4 The p-adic number field

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## 2.4 THE $p$ -ADIC NUMBER FIELD

Let  $p$  be prime. We shall use  $\mathbf{Z}_p$  to represent the  $p$ -adic integers, and  $\mathbf{Q}_p$  the  $p$ -adic rationals. Let  $|\cdot|_p$  denote the  $p$ -adic absolute value on  $\mathbf{Q}_p$ , normalized so that  $|p|_p = p^{-1}$ . Let  $\overline{\mathbf{Q}}_p$  be the algebraic closure of  $\mathbf{Q}_p$ . The absolute value on  $\mathbf{Q}_p$  extends uniquely to  $\overline{\mathbf{Q}}_p$ , however  $\overline{\mathbf{Q}}_p$  is not complete with respect to the absolute value. Let  $\mathbf{C}_p$  be the completion of  $\overline{\mathbf{Q}}_p$  with respect to this absolute value. Then the absolute value extends to  $\mathbf{C}_p$ , and  $\overline{\mathbf{Q}}_p$  is dense in  $\mathbf{C}_p$ . We also have  $\mathbf{C}_p$  algebraically closed. Furthermore, on  $\mathbf{C}_p$  the absolute value is non-Archimedean, and so

$$|a + b|_p \leq \max\{|a|_p, |b|_p\}$$

for any  $a, b \in \mathbf{C}_p$ . Note that the two fields  $\mathbf{C}$  and  $\mathbf{C}_p$  are algebraically isomorphic, and any one of the two can be embedded in the other. We denote two particular subrings of  $\mathbf{C}_p$  in the following manner

$$\mathfrak{o} = \{a \in \mathbf{C}_p : |a|_p \leq 1\}, \quad \mathfrak{p} = \{a \in \mathbf{C}_p : |a|_p < 1\}.$$

Then  $\mathfrak{p}$  is a maximal ideal of  $\mathfrak{o}$ . If  $\tau \in \mathbf{C}_p$  such that  $|\tau|_p \leq |p|_p^s$ , where  $s \in \mathbf{Q}$ , then  $\tau \in p^s \mathfrak{o}$ , and so we shall also write this as  $\tau \equiv 0 \pmod{p^s \mathfrak{o}}$ .

Any  $n \in \mathbf{Z}$ ,  $n > 0$ , can be uniquely expressed in the form  $n = \sum_{m=0}^k a_m p^m$ , where  $a_m \in \mathbf{Z}$ ,  $0 \leq a_m \leq p - 1$ , for  $m = 0, 1, \dots, k$ , and  $a_k \neq 0$ . For such  $n$ , we define

$$s_p(n) = \sum_{m=0}^k a_m,$$

the sum of the  $p$ -adic digits of  $n$ , and also define  $s_p(0) = 0$ . For any  $n \in \mathbf{Z}$ , let  $v_p(n)$  be the highest power of  $p$  dividing  $n$ . This function is additive, and relates to the function  $s_p(n)$  by means of the identity

$$(8) \quad v_p(n!) = \frac{n - s_p(n)}{p - 1},$$

which holds for all  $n \geq 0$ . Note that for  $n \geq 1$  this implies that

$$v_p(n!) \leq \frac{n - 1}{p - 1}.$$

The definition of this function can be extended to all of  $\mathbf{Q}$  by taking  $v_p(1/n) = -v_p(n)$ .

Throughout we let  $q = 4$  if  $p = 2$ , and  $q = p$  otherwise. Note that there exist  $\phi(q)$  distinct solutions, modulo  $q$ , to the equation  $x^{\phi(q)} - 1 = 0$ , and each solution must be congruent to one of the values  $a \in \mathbf{Z}$ , where  $1 \leq a \leq q$ ,

$(a, p) = 1$ . Thus, by Hensel's Lemma, given  $a \in \mathbf{Z}$  with  $(a, p) = 1$ , there exists a unique  $\omega(a) \in \mathbf{Z}_p$ , where  $\omega(a)^{\phi(q)} = 1$ , such that

$$\omega(a) \equiv a \pmod{q\mathbf{Z}_p}.$$

Letting  $\omega(a) = 0$  for  $a \in \mathbf{Z}$  such that  $(a, p) \neq 1$ , we see that  $\omega$  is actually a Dirichlet character, called the Teichmüller character, having conductor  $f_\omega = q$ . Let us define

$$\langle a \rangle = \omega^{-1}(a)a.$$

Then  $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$ . For  $p \geq 3$ ,  $\lim_{n \rightarrow \infty} a^{p^n} = \omega(a)$ , since  $a^{p^n} \equiv a \pmod{p}$  and  $a^{p^n(p-1)} \equiv 1 \pmod{p^{n+1}}$ .

For our purposes we shall need to make a slight extension of the definition of the Teichmüller character  $\omega$ . If  $t \in \mathbf{C}_p$  such that  $|t|_p \leq 1$ , then for any  $a \in \mathbf{Z}$ ,  $a + qt \equiv a \pmod{q\mathbf{Z}}$ . Thus we define

$$\omega(a + qt) = \omega(a)$$

for these values of  $t$ . We also define

$$\langle a + qt \rangle = \omega^{-1}(a)(a + qt)$$

for such  $t$ .

Fix an embedding of the algebraic closure of  $\mathbf{Q}$ ,  $\overline{\mathbf{Q}}$ , into  $\mathbf{C}_p$ . We may then consider the values of a Dirichlet character  $\chi$  as lying in  $\mathbf{C}_p$ . For  $n \in \mathbf{Z}$  we define the product  $\chi_n = \chi\omega^{-n}$  in the sense of the product of characters. This implies that  $f_{\chi_n} | f_\chi q$ . However, since we can write  $\chi = \chi_n\omega^n$ , we also have  $f_\chi | f_{\chi_n}q$ . Thus  $f_\chi$  and  $f_{\chi_n}$  differ by a factor that is a power of  $p$ . In fact, either  $f_{\chi_n}/f_\chi \in \mathbf{Z}$  and divides  $q$ , or  $f_\chi/f_{\chi_n} \in \mathbf{Z}$  and divides  $q$ .

Let  $\mathbf{Q}_p(\chi)$  denote the field generated over  $\mathbf{Q}_p$  by all values  $\chi(a)$ ,  $a \in \mathbf{Z}$ . In this context we can state the following, found in [13] (pp. 14–15).

LEMMA 2.2. *In the field  $\mathbf{Q}_p(\chi)$ , for all  $n \in \mathbf{Z}$ ,  $n \geq 0$ ,*

$$B_{n,\chi} = \frac{1}{n+1} \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} (B_{n+1,\chi}(p^h f_\chi) - B_{n+1,\chi}(0)).$$

From this we can obtain

LEMMA 2.3. Let  $\tau \in \mathbf{C}_p$ . In the field  $\mathbf{Q}_p(\chi, \tau)$ , for all  $n \in \mathbf{Z}$ ,  $n \geq 0$ ,

$$B_{n,\chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a + \tau)^n.$$

*Proof.* By applying Lemma 2.2 to (4), we obtain

$$B_{n,\chi} = \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi(a)a^n.$$

Therefore, by (2),

$$\begin{aligned} B_{n,\chi_n}(\tau) &= \sum_{m=0}^n \binom{n}{m} \tau^{n-m} \lim_{h \rightarrow \infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a)a^m \\ &= \lim_{h \rightarrow \infty} \frac{1}{p^h f_{\chi_n}} \sum_{a=1}^{p^h f_{\chi_n}} \chi_n(a) \sum_{m=0}^n \binom{n}{m} \tau^{n-m} a^m. \end{aligned}$$

Since  $f_\chi$  and  $f_{\chi_n}$  differ by a factor that is a power of  $p$ , we must have

$$B_{n,\chi_n}(\tau) = \lim_{h \rightarrow \infty} \frac{1}{p^h f_\chi} \sum_{a=1}^{p^h f_\chi} \chi_n(a)(a + \tau)^n,$$

and the proof is complete.  $\square$

## 2.5 $p$ -ADIC FUNCTIONS

Let  $K$  be an extension of  $\mathbf{Q}_p$  contained in  $\mathbf{C}_p$ . An infinite series  $\sum_{n=0}^{\infty} a_n$ ,  $a_n \in K$ , converges in  $K$  if and only if  $|a_n|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $K[[x]]$  be the algebra of formal power series in  $x$ . Then it follows that a power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,$$

in  $K[[x]]$ , converges at  $x = \xi$ ,  $\xi \in \mathbf{C}_p$ , if and only if  $|a_n \xi^n|_p \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore whenever a power series  $A(x)$  converges at some  $\xi_0 \in \mathbf{C}_p$ , then it must converge at all  $\xi \in \mathbf{C}_p$  such that  $|\xi|_p \leq |\xi_0|_p$ . The following result, for double series in  $K$ , can be found in [8].