4.3 Relating $L_p(s, t; \lambda)$ to some finite sums

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With this result, we can derive a more general power series expansion of $L_p(s,t;\chi)$.

THEOREM 4.7. *Let* $t \in \mathbf{C}_p$, $|t|_p \le 1$, *and* $s \in \mathfrak{D}$, *except* $s \neq 1$ *if* $\chi = 1$. *Then for* $\alpha \in \mathbf{C}_p$, $|\alpha|_p \le 1$,

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \binom{-s}{m} q^m (t-\alpha)^m L_p(s+m,\alpha;\chi_m) .$$

REMARK. Note that Theorem 4.5 is the case of $\alpha = 0$ here.

Proof. It follows from the Taylor series expansion of $L_p(s,t;\chi)$ in the variable t about α (see Proposition 2.6) that we can write $L_p(s,t;\chi)$ in the form

$$L_p(s,t;\chi) = \sum_{m=0}^{\infty} \beta_m (t-\alpha)^m,$$

where

$$\beta_m = \frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s,t;\chi)\bigg|_{t=\alpha} .$$

From Lemma 4.6

$$\frac{1}{m!} \frac{\partial^m}{\partial t^m} L_p(s,t;\chi) = \binom{-s}{m} q^m L_p(s+m,t;\chi_m),$$

and so

$$\beta_m = \binom{-s}{m} q^m L_p(s+m,\alpha;\chi_m),$$

completing the proof. $\square$

4.3 RELATING $L_p(s,t;\chi)$ TO SOME FINITE SUMS

From (4) it becomes obvious that the generalized Bernoulli polynomials have a considerable significance in regard to sums of consecutive nonnegative integers, each raised to the same power, itself a nonnegative integer. The following illustrates how this can be extended with the use of $L_p(s,t;\chi)$.

For the character χ, let $F_0 = \operatorname{lcm}(f_\chi, q)$. Then $f_{\chi_n} \mid F_0$ for each $n \in \mathbf{Z}$. Also, let F be a positive multiple of $pq^{-1}F_0$.

THEOREM 4.8. *Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, t+F; \chi) - L_p(s, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a + qt \rangle^{-s} . \tag{23}$$

Proof. Let $t \in \mathbf{C}_p$, $|t|_p \leq 1$, and let $n \in \mathbf{Z}$, $n \geq 1$. Then from (18),

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = -\frac{1}{n} (b_n(t+F) - b_n(t)) .$$

Now, (19) implies

$$\begin{aligned} b_n(t+F) - b_n(t) &= \left(B_{n,\chi_n}(q(t+F)) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1} q(t+F)\right)\right) \\ &\quad - \left(B_{n,\chi_n}(qt) - \chi_n(p) p^{n-1} B_{n,\chi_n}\left(p^{-1} qt\right)\right) \\ &= \left(B_{n,\chi_n}(q(t+F)) - B_{n,\chi_n}(qt)\right) \\ &\quad - \chi_n(p) p^{n-1} \left(B_{n,\chi_n}\left(p^{-1} q(t+F)\right) - B_{n,\chi_n}\left(p^{-1} qt\right)\right) . \end{aligned}$$

Thus, by (4), we can write

$$\begin{aligned} &b_n(t+F) - b_n(t) \\ &\quad = n \sum_{a=1}^{qF} \chi_n(a)(a+qt)^{n-1} - n\chi_n(p) p^{n-1} \sum_{a=1}^{p^{-1}qF} \chi_n(a)(a+p^{-1}qt)^{n-1} \\ &\quad = n \sum_{a=1}^{qF} \chi_n(a)(a+qt)^{n-1} - n \sum_{\substack{a=1 \\ p|a}}^{qF} \chi_n(a)(a+qt)^{n-1} . \end{aligned}$$

Therefore,

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_n(a)(a+qt)^{n-1} .$$

Now, $\chi_n = \chi_1 \omega^{-(n-1)}$, so that

$$\begin{aligned} \chi_n(a)(a+qt)^{n-1} &= \chi_1(a) \omega^{-(n-1)}(a)(a+qt)^{n-1} \\ &= \chi_1(a) \langle a+qt \rangle^{n-1} . \end{aligned}$$

Thus

$$L_p(1-n, t+F; \chi) - L_p(1-n, t; \chi) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a+qt \rangle^{n-1} ,$$

and (23) holds for all $s = 1 - n$, where $n \in \mathbf{Z}$, $n \geq 1$. Therefore, since the negative integers have 0 as a limit point, Lemma 2.5 implies that Theorem 4.8 holds for all s in any neighborhood about 0 common to the domains of the functions on either side of (23).

It is obvious that the domains, in the variable s, of the functions on the left of (23) contain $\mathfrak{D}$, except $s \neq 1$ when $\chi = 1$. Consider now the function

$$-\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{-s} = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a+qt\rangle^{-1}\langle a+qt\rangle^{1-s}.$$

Since it consists of a finite sum of functions of the form $\langle a+qt\rangle^{1-s}$, where $a \in \mathbf{Z}$, $(a,p) = 1$, we need only show that each such function is analytic on $\mathfrak{D}$, and the proof will be complete.

The quantity $\langle a+qt\rangle^{1-s}$ can be written as

$$\langle a+qt\rangle^{1-s} = \exp\left((1-s)\log\langle a+qt\rangle\right),$$

and by (9), the Taylor series expansion of the exponential function,

$$\langle a+qt\rangle^{1-s} = \sum_{m=0}^{\infty} \frac{1}{m!}(1-s)^m \left(\log\langle a+qt\rangle\right)^m.$$

Since $\langle a+qt\rangle \equiv 1 \pmod{q\mathfrak{o}}$ for $a \in \mathbf{Z}$, $(a,p) = 1$, and $t \in \mathbf{C}_p$, $|t|_p \leq 1$, we must also have $\log\langle a+qt\rangle \equiv 0 \pmod{q\mathfrak{o}}$ for such a and t. Thus

$$\left|\frac{1}{m!}(1-s)^m\left(\log\langle a+qt\rangle\right)^m\right|_p \leq \left|\frac{1}{m!}q^m(s-1)^m\right|_p$$

for all m. By (8) we can write

$$\begin{aligned}\left|\frac{1}{m!}q^m(s-1)^m\right|_p &\leq \left|p^{-m/(p-1)}q^m(s-1)^m\right|_p\\ &= \left|p^{-1/(p-1)}q(s-1)\right|_p^m.\end{aligned}$$

Thus if

$$\left|p^{-1/(p-1)}q(s-1)\right|_p < 1,$$

then

$$\left|\frac{1}{m!}(1-s)^m\left(\log\langle a+qt\rangle\right)^m\right|_p \to 0$$

as $m \to \infty$. So whenever $|s-1|_p < |p|_p^{1/(p-1)}|q|_p^{-1}$, meaning that $s \in \mathfrak{D}$, we have convergence for the power series. Therefore, the functions on either side of (23) have domains that contain $\mathfrak{D}$, except possibly for $s = 1$ when $\chi = 1$, and the theorem must hold. □

COROLLARY 4.9. *Let $s \in \mathfrak{D}$, except $s \neq 1$ if $\chi = 1$. Then*

$$L_p(s, F; \chi) = L_p(s; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-s}.$$

Proof. This follows from Theorem 4.8 since $L_p(s, 0; \chi) = L_p(s; \chi)$ for any character χ. □

We shall now consider how Corollary 4.9 can be utilized to derive a collection of congruences related to the generalized Bernoulli polynomials. Let Δ_c denote the forward difference operator, $\Delta_c x_n = x_{n+c} - x_n$. Repeated application of this operator can be expressed in the form

$$\Delta_c^k x_n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} x_{n+mc}.$$

Recall that $F_0 = \operatorname{lcm}(f_\chi, q)$. For $n \in \mathbf{Z}$, $n \geq 1$, denote

$$\beta_{n,\chi}(t) = -\frac{1}{n}\left(B_{n,\chi_n}(qt) - \chi_n(p)p^{n-1}B_{n,\chi_n}\left(p^{-1}qt\right)\right).$$

This is the polynomial structure that we utilized with respect to generalizing the p-adic L-functions. We will incorporate this structure in an extension of the Kummer congruences, but the results that we derive will be without restriction on either χ or p.

THEOREM 4.10. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then the quantity $q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$, and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.*

Proof. Since Δ_c is a linear operator, Corollary 4.9 implies that

$$\Delta_c^k L_p(1-n, F; \chi) = \Delta_c^k L_p(1-n; \chi) - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\Delta_c^k\langle a\rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Thus

$$\Delta_c^k\beta_{n,\chi}(F) - \Delta_c^k\beta_{n,\chi}(0) = -\sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-1}\Delta_c^k\langle a\rangle^{n}.$$

Note that

$$\Delta_c^k \langle a \rangle^n = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} \langle a \rangle^{n+mc} = \langle a \rangle^n \left(\langle a \rangle^c - 1 \right)^k . \tag{24}$$

Now, $\langle a \rangle \equiv 1 \pmod{q\mathbf{Z}_p}$, which implies that $\langle a \rangle^c \equiv 1 \pmod{q\mathbf{Z}_p}$, and thus

$$\Delta_c^k \langle a \rangle^n \equiv 0 \pmod{q^k \mathbf{Z}_p} .$$

Therefore

$$\Delta_c^k \beta_{n,\chi}(F) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]} ,$$

and so $q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Also, since $\langle a \rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$,

$$q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) = - \sum_{\substack{a=1 \\ (a,p)=1}}^{qF} \chi_1(a) \langle a \rangle^{n-1} \left(\frac{\langle a \rangle^c - 1}{q} \right)^k \tag{25}$$

implies that the value of $q^{-k}\Delta_c^k \beta_{n,\chi}(F) - q^{-k}\Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n.

Let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Since the set of positive integers in $pq^{-1}F_0\mathbf{Z}$ is dense in $pq^{-1}F_0\mathbf{Z}_p$, there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. Now, $\beta_{n,\chi}(t)$ is a polynomial, which implies that $\beta_{n,\chi}(\tau_i) \to \beta_{n,\chi}(\tau)$. Therefore

$$\lim_{i \to \infty} \left(\Delta_c^k \beta_{n,\chi}(\tau_i) - \Delta_c^k \beta_{n,\chi}(0) \right) = \Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) .$$

The left side of this equality is 0 modulo $q^k\mathbf{Z}_p[\chi]$, which implies that

$$\Delta_c^k \beta_{n,\chi}(\tau) - \Delta_c^k \beta_{n,\chi}(0) \equiv 0 \pmod{q^k \mathbf{Z}_p[\chi]} ,$$

and so $q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi]$. Furthermore, for n' a positive integer,

$$\begin{aligned} &\lim_{i \to \infty} \left(\left(q^{-k}\Delta_c^k \beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \right) - \left(q^{-k}\Delta_c^k \beta_{n',\chi}(\tau_i) - q^{-k}\Delta_c^k \beta_{n',\chi}(0) \right) \right) \\ &\quad = \left(\left(q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0) \right) - \left(q^{-k}\Delta_c^k \beta_{n',\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n',\chi}(0) \right) \right) . \end{aligned}$$

Since $\tau_i \in pq^{-1}F_0\mathbf{Z}$ for each i, the quantity on the left must also be 0 modulo $q\mathbf{Z}_p[\chi]$. Therefore the value of $q^{-k}\Delta_c^k \beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k \beta_{n,\chi}(0)$ modulo $q\mathbf{Z}_p[\chi]$ is independent of n. □

THEOREM 4.11. *Let n, c, k, and k' be positive integers with $k \equiv k'$ $(\text{mod } p-1)$, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \leq |pq^{-1}F_0|_p$. Then*

$$q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \ (\text{mod } p\mathbf{Z}_p[\chi]).$$

Proof. Let k and k' be positive integers such that $k \equiv k'$ $(\text{mod } p-1)$. Without loss of generality, we can assume that $k \geq k'$. From (25),

$$\left(q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right) - \left(q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)\right) \\ = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c-1}{q}\right)^k + \sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c-1}{q}\right)^{k'} \\ = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{n-1}\left(\frac{\langle a\rangle^c-1}{q}\right)^{k'}\left(\left(\frac{\langle a\rangle^c-1}{q}\right)^{k-k'}-1\right),$$

where F is a positive multiple of $pq^{-1}F_0$. If a is such that

$$\langle a\rangle^c - 1 \not\equiv 0 \ (\text{mod } pq\mathbf{Z}_p),$$

then

$$\left(\frac{\langle a\rangle^c-1}{q}\right)^{k-k'} - 1 \equiv 0 \ (\text{mod } p\mathbf{Z}_p),$$

since $k-k' \equiv 0$ $(\text{mod } p-1)$. Thus

$$q^{-k}\Delta_c^k\beta_{n,\chi}(F) - q^{-k}\Delta_c^k\beta_{n,\chi}(0) \\ \equiv q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(F) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0) \ (\text{mod } p\mathbf{Z}_p[\chi]).$$

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$. Then there exists a sequence $\{\tau_i\}_{i=1}^{\infty}$ in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. Consider

$$\lim_{i\to\infty}\left((q^{-k}\Delta_c^k\beta_{n,\chi}(\tau_i) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)) - (q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau_i) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0))\right) \\ = \left(q^{-k}\Delta_c^k\beta_{n,\chi}(\tau) - q^{-k}\Delta_c^k\beta_{n,\chi}(0)\right) - \left(q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(\tau) - q^{-k'}\Delta_c^{k'}\beta_{n,\chi}(0)\right).$$

Since the left side of this equality must be 0 modulo $p\mathbf{Z}_p[\chi]$, the theorem must hold. $\square$

THEOREM 4.12. *Let n, c, and k be positive integers, and let $\tau \in \mathbf{Z}_p$ such that $|\tau|_p \le |pq^{-1}F_0|_p$. Then the quantity*

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi],$$

and, modulo $q\mathbf{Z}_p[\chi]$, is independent of n.

Proof. We are once again working with a linear operator, so Corollary 4.9 implies that

$$\binom{q^{-1}\Delta_c}{k}L_p(1-n,F;\chi) = \binom{q^{-1}\Delta_c}{k}L_p(1-n;\chi) - \sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n-1},$$

where F is a positive multiple of $pq^{-1}F_0$. Then

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) = -\sum_{\substack{a=1\\(a,p)=1}}^{qF} \chi_1(a)\langle a\rangle^{-1}\binom{q^{-1}\Delta_c}{k}\langle a\rangle^{n}.$$

Utilizing (15), we can write

$$\begin{aligned}\binom{q^{-1}\Delta_c}{k}\langle a\rangle^n &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)q^{-m}\Delta_c^m\langle a\rangle^n\\ &= \frac{1}{k!}\sum_{m=0}^{k} s(k,m)q^{-m}\langle a\rangle^n\left(\langle a\rangle^c - 1\right)^m,\end{aligned}$$

which follows from (24). This can then be rewritten as

$$\binom{q^{-1}\Delta_c}{k}\langle a\rangle^n = \langle a\rangle^n\binom{q^{-1}(\langle a\rangle^c - 1)}{k}.$$

Since $q^{-1}(\langle a\rangle^c - 1) \in \mathbf{Z}_p$ for each $a \in \mathbf{Z}$ with $(a,p) = 1$, we see that

$$\langle a\rangle^n\binom{q^{-1}(\langle a\rangle^c - 1)}{k} \in \mathbf{Z}_p.$$

This then implies that

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(F) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0) \in \mathbf{Z}_p[\chi].$$

Furthermore, since $\langle a\rangle^n \equiv 1 \pmod{q\mathbf{Z}_p}$, the value of this quantity modulo $q\mathbf{Z}_p[\chi]$ is independent of n.

Now let $\tau \in pq^{-1}F_0\mathbf{Z}_p$, and let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence in $pq^{-1}F_0\mathbf{Z}$, with $\tau_i > 0$ for each i, such that $\tau_i \to \tau$. We are working with polynomials, so that

$$\lim_{i\to\infty}\left(\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau_i) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)\right) = \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0),$$

which must be in $\mathbf{Z}_p[\chi]$ since the limit of any sequence in $\mathbf{Z}_p[\chi]$ must also be in $\mathbf{Z}_p[\chi]$. Now let n' be a positive integer, and consider

$$\lim_{i\to\infty}\left(\left(\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n,\chi}(\tau_i)-\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n,\chi}(0)\right)-\left(\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n',\chi}(\tau_i)-\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n',\chi}(0)\right)\right)$$
$$=\left(\left(\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n,\chi}(\tau)-\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n,\chi}(0)\right)-\left(\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n',\chi}(\tau)-\left(\tbinom{q^{-1}\Delta_c}{k}\right)\beta_{n',\chi}(0)\right)\right).$$

The quantity on the left must be 0 modulo $q\mathbf{Z}_p[\chi]$, which implies that the value of

$$\binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(\tau) - \binom{q^{-1}\Delta_c}{k}\beta_{n,\chi}(0)$$

modulo $q\mathbf{Z}_p[\chi]$ is independent of n. $\square$

4.4 Generalized Bernoulli power series

In [9] we find a definition of ordinary Bernoulli numbers of negative index, B_{-n}, where $n \in \mathbf{Z}$, $n \geq 1$, in the field $\mathbf{Q}_p$, given by

$$B_{-n} = \lim_{k\to\infty} B_{\phi(p^k)-n}, \tag{26}$$

where the limit is taken in a p-adic sense. Note that $\phi(p^k) \to 0$ in $\mathbf{Z}_p$ as $k \to \infty$. Since $|B_m|_p$ is bounded for all $m \in \mathbf{Z}$, $m \geq 0$, we must have

$$\begin{aligned} B_{-n} &= \lim_{k\to\infty}\left(1 - p^{\phi(p^k)-n-1}\right) B_{\phi(p^k)-n} \\ &= \lim_{k\to\infty} -\left(\phi\left(p^k\right) - n\right) L_p\left(1 - \left(\phi\left(p^k\right) - n\right); \omega^{-n}\right) \\ &= nL_p\left(n+1; \omega^{-n}\right). \end{aligned}$$

implying that the limit exists and can be described in familiar terms.

Recall that $B_m = 0$ for any odd $m \in \mathbf{Z}$, $m \geq 3$. Thus (26) implies that $B_{-n} = 0$ for any odd $n \in \mathbf{Z}$, $n \geq 1$. Furthermore, we have the following: