5. Which pairs of curve diagrams determine the same ordering?

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In this section we define an equivalence relation of curve diagrams which we call *loose isotopy*. We give a simple algorithm to decide whether or not two given curve diagrams are loosely isotopic. We prove that two curve diagrams determine the same ordering if and only if they are loosely isotopic. Moreover, the quotient of the set of loose isotopy classes of curve diagrams under the natural action of B_n is finite; we deduce that for fixed $n \geqslant 2$ there is only a finite number of conjugacy classes of orderings arising from curve diagrams.

DEFINITION 5.1. Let $\mathcal{C}$ denote the space of all curve diagrams, equipped with the natural topology (the subset topology from the space of all mappings of $n-1$ arcs into D_n). We define *loose isotopy* to be the equivalence relation on $\mathcal{C}$ generated by the following two types of equivalence:

(1) *Continuous deformation:* two curve diagrams are equivalent if they lie in the same path component of $\mathcal{C}$.

(2) *Pulling loops around punctures tight:* if some final segment of the curve Γ_i say cuts out a disk with one puncture from D_n, then this final segment can be pulled tight, so as to make Γ_i end in the puncture.

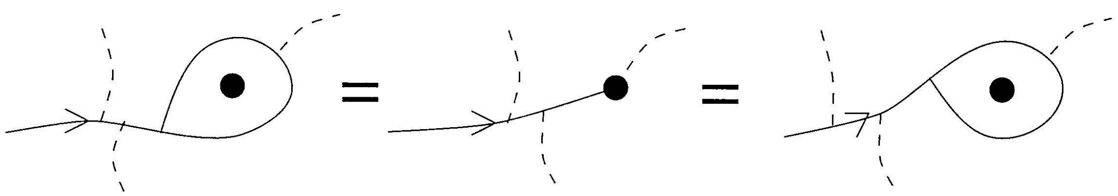

FIGURE 4
Pulling loops around punctures tight

Equivalence (2) is illustrated in Figure 4; here the dashed lines indicate any number of arcs of index greater than i which start on Γ_i. Equivalence (1) says that one is allowed to deform the diagram, to slide starting points of arcs along the union of all previous arcs, including their start and end points, and even across punctures, if they are the end points of some previous arcs. Similarly, end points of arcs are allowed to slide across the union of all "previous points of the diagram".

In order to get a feel for the meaning of this definition, the reader may want to prove that the equality signs in Figure 2 represent loose isotopies.

THEOREM 5.2. (a) *Two curve diagrams determine the same ordering of B_n if and only if they are loosely isotopic.*

(b) *There is an algorithm to decide whether or not two curve diagrams Γ and Δ are loosely isotopic.*

Proof. For the implication "$\Leftarrow$" of (a) we have to prove that loosely isotopic diagrams define the same ordering. The only nonobvious claim here is that the ordering is invariant under the "pulling tight" procedure.

In order to prove this, we consider a curve diagram Γ' with j arcs, the i^{th} of which is a loop (i.e. the end point equals the start point) which encloses exactly one puncture. We consider in addition the curve diagram Γ which is obtained from Γ' by squashing the curve Γ'_i to an arc from the starting point of Γ'_i to the enclosed puncture, much as in Figure 4. Let φ and ψ be two nonisotopic homeomorphisms, and more precisely assume that $\varphi >_{\Gamma} \psi$. Our aim is to prove that $\varphi >_{\Gamma'} \psi$. If $\varphi(\Gamma_{0\cup\cdots\cup i-1})$ and $\psi(\Gamma_{0\cup\cdots\cup i-1})$ are already nonisotopic then this is obvious since the first $i-1$ arcs of Γ and Γ' coincide. On the other hand, if $\varphi(\Gamma_{0\cup\cdots\cup i})$ and $\psi(\Gamma_{0\cup\cdots\cup i})$ are isotopic (and the difference between φ and ψ only shows up on arcs of higher index), then after an isotopy the first i arcs of $\varphi(\Gamma')$ and $\psi(\Gamma')$ coincide as well, and the result follows easily. Finally in the critical case, when the first difference occurs on the i^{th} arc of Γ, we have the two arcs $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$ which are reduced with respect to each other, with $\varphi(\Gamma_i)$ setting off more to the left. The crucial observation is now that the boundary curves of sufficiently small regular neighbourhoods of the two curves are isotopic to $\varphi(\Gamma'_i)$ respectively $\psi(\Gamma'_i)$ *and* reduced with respect to each other – see Figure 5. It is now clear that $\varphi(\Gamma'_i)$ also sets off more to the left than $\psi(\Gamma'_i)$. This completes the proof of implication "$\Leftarrow$" of (a).

We shall now explicitly describe the algorithm promised in (b), and prove the implication "$\Rightarrow$" of (a) along the way. The proof is by induction on n. For the case $n = 2$ we note that any two total curve diagrams (with one arc) are loosely isotopic. Thus there are only two loose isotopy classes of curve diagrams: the empty diagram and the one with one arc. The empty diagram induces the trivial ordering, whereas the diagram with one arc induces the ordering $\sigma_1^k > \sigma_1^l \iff k > l$. So the desired algorithm consists just of counting the number of arcs, and non loosely isotopic curve diagrams do indeed induce different orderings.

Now suppose that $n \geq 3$, that the result is true for disks with fewer than n punctures, and that we want to compare two curve diagrams $\Gamma_0, \ldots, \Gamma_j$ and $\Delta_0, \ldots, \Delta_{j'}$ in D_n, with $j, j' \leq n-1$. The arc Γ_1 ends either on ∂D_n, or in

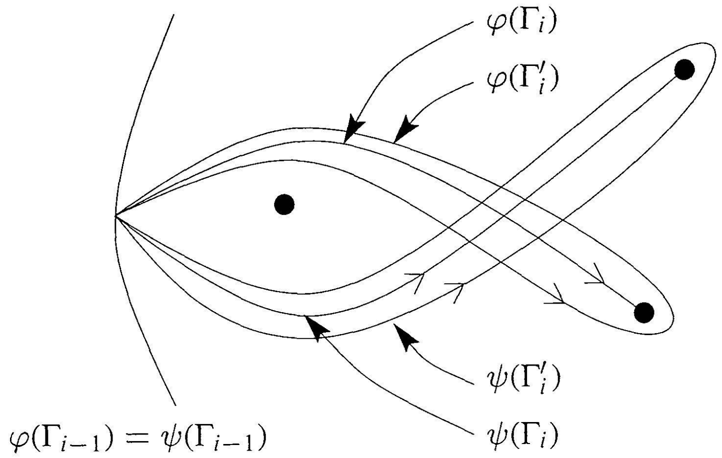

FIGURE 5

Proof that $\varphi >_\Gamma \psi \Rightarrow \varphi >_{\Gamma'} \psi$ – the critical case where the first difference between φ and ψ occurs on the arc which is being pulled tight

the interior of Γ_1 itself, or in a puncture. In the first two cases $D_n \backslash \Gamma_1$ has precisely two path components. At most one of them can contain only one puncture; if one of them does, we pull Γ_1 tight around it. If both components of $D_n \backslash \Gamma_1$ contain more than one puncture and if Γ_1 ends on itself, then we slide the end point of Γ_1 back along Γ_1, across its starting point, and into $\Gamma_0 = \partial D_n$. There are now two possibilities left: either Γ_1 is an embedded arc connecting the boundary to a puncture (Γ_1 is *nonseparating*), or it is an embedded arc connecting two boundary points, cutting D_n into two pieces, each of which has at least two punctures in its interior (Γ_1 is *separating*). We repeat this procedure for Δ_1. There are now four cases:

(1) It may be that Γ_1 is separating, while Δ_1 is not (or vice versa).

(2) It is possible that Γ_1 and Δ_1 are both nonseparating but are not isotopic with starting points sliding in ∂D_n (a criterion which is easy to check algorithmically).

(3) It is possible that Γ_1 and Δ_1 are both separating but are not isotopic as oriented arcs, with starting and end points sliding in ∂D_n (a criterion which is equally easy to check algorithmically).

CLAIM. *In these first three cases the orderings defined by* Γ *and* Δ *do not coincide, and* Γ *and* Δ *are not loosely isotopic.*

We only need to prove the first part of the claim, the second one follows by the implication "$\Leftarrow$" of Theorem 5.2(a). We first treat the following pathological situation: if, in case (3) above, Γ_1 and Δ_1 are isotopic to each other, but with opposite orientations, then a homeomorphism of the type

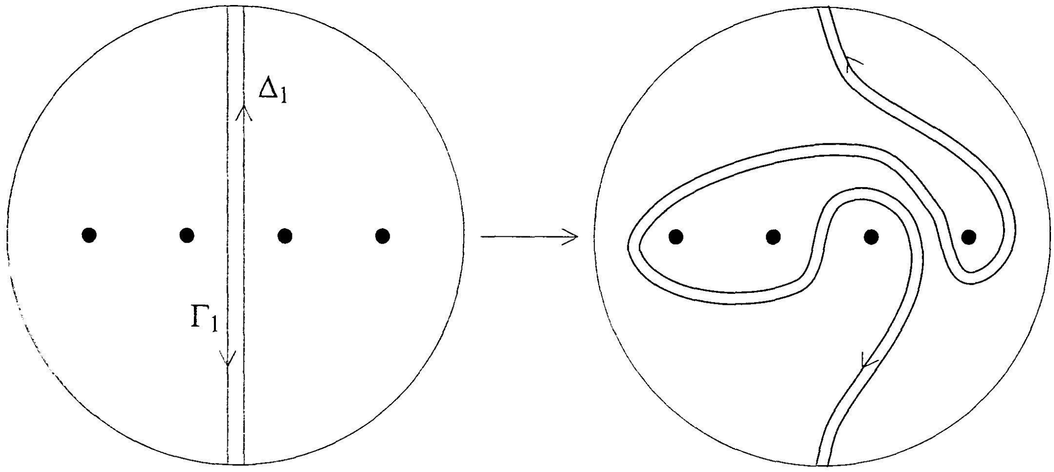

FIGURE 6

A homeomorphism which distinguishes the Γ- and Δ-orderings

indicated in Figure 6 is positive in the ordering defined by Γ, but negative in the Δ-ordering. In all other situations allowed by (1), (2) and (3), there exists a simple closed curve τ in D_n which is disjoint from Γ_1, but intersects every arc isotopic to Δ_1. (Consider, for instance, a regular neighbourhood of $\partial D_n \cup \Gamma_1$ in D_n. If Γ_1 is nonseparating then its boundary curve has this property; if Γ_1 is separating then at least one of the two boundary curves has.) We denote by $T\colon D_n \to D_n$ the positive Dehn twist along τ. The map T leaves Γ_1 invariant, while the arc $T(\Delta_1)$ is "more to the left" than Δ_1 (to see this, reduce the two arcs by making them geodesic, and apply Proposition 2.4).

Similarly, there exists a curve τ' which is disjoint from Δ_1, but not from any arc in the isotopy class of Γ_1. Then ${T'}^{-1}$ sends $T(\Delta_1)$ more to the right, but not very far: ${T'}^{-1} \circ T(\Delta_1)$ is still to the left of the arc Δ_1, which is fixed by ${T'}^{-1}$; and ${T'}^{-1}$ sends Γ_1 to the right, as well. Thus, in summary, the composition ${T'}^{-1} \circ T$ sends Δ_1 more to the left but Γ_1 more to the right, so that ${T'}^{-1} \circ T \in B_n$ is negative in the ordering determined by Γ, but positive in the Δ-ordering. This proves the claim. (One may find simpler proofs, but this one will be useful in Section 7.)

(4) The remaining possibility is that Γ_1 and Δ_1 *can* be made to coincide by isotopies which need not be fixed on ∂D_n. Such isotopies can be extended to loose isotopies of Γ or Δ.

To summarize, we can algorithmically decide whether or not there is a loose isotopy which makes Γ_1 and Δ_1 coincide. If the answer is NO (cases (1)–(3)), then Γ and Δ are not loosely isotopic, and the orderings defined by Γ and Δ do not coincide. In this case, the implication "$\Rightarrow$" of 5.2(a) is true. If the answer is YES (case (4)), then $D_n \backslash \Gamma_1 = D_n \backslash \Delta_1$ has either one or two path components, each of which is a disk with at most $n-1$ punctures. Moreover, the arcs $\Gamma_2, \dots, \Gamma_j$ form curve diagrams in these disks (with some indices missing in each curve diagram, if the arcs are distributed among two disks), and similarly for $\Delta_2, \dots, \Delta_{j'}$. Finally, the following conditions are equivalent:

(i) Γ and Δ are loosely isotopic,

(ii) in each path component of $D_n \setminus \Gamma_1 = D_n \setminus \Delta_1$ there is a loose isotopy between the diagrams made up of the remaining arcs of Γ respectively Δ,

(iii) the orderings of $Fix(\Gamma_1) \subseteq B_n$ induced by Γ and Δ coincide, where $Fix(\Gamma_1)$ denotes the subgroup whose elements have support disjoint from Γ_1,

(iv) the orderings of B_n defined by Γ and Δ coincide.

The equivalences between (i) and (ii), and between (iii) and (iv) are clear, whereas the equivalence of (ii) and (iii) follows from the induction hypothesis. Also by the induction hypothesis, we can decide algorithmically whether or not (ii) holds. This proves the theorem in case (4). □

We recall that for any ordering "$<$" of B_n, and every element $\rho \in B_n = \mathcal{MCG}(D_n)$, one can construct an ordering "$<_\rho$", by defining $\varphi <_\rho \psi :\Longleftrightarrow \varphi\rho < \psi\rho$, and we call $<_\rho$ "the ordering $<$ conjugated by ρ". We observe that if $<$ is induced by a curve diagram Γ, then $<_\rho$ is induced by the curve diagram $\rho(\Gamma)$. Thus two curve diagrams Γ and Δ induce conjugate orderings if and only if Γ and Δ are in the same orbit under the natural action of B_n on the set of loose isotopy classes of curve diagrams.

PROPOSITION 5.3. *Let M_n denote the number of conjugacy classes of total orderings of B_n arising from curve diagrams. Then M_n can be calculated by the following recursive formula*

$$M_2 = 1 \quad \textit{and} \quad M_n = M_{n-1} + \sum_{k=2}^{n-2} \binom{n-2}{k-1} M_k \, M_{n-k} \, .$$

REMARK. In order to avoid confusion, we recall our orientation convention: we are insisting that "more to the left" means "larger". It is for this reason that there is only one ordering of $B_2 = \mathbf{Z}$, not two, as one might expect.

Proof. We shall count the orbits of the set of loose isotopy classes of total curve diagrams under the action of B_n. The case $n = 2$ is clear, since there is only one loose isotopy class of curve diagrams. Now suppose inductively that the formula is true for up to $n-1$ strings.

For every total curve diagram in D_n there are two possibilities:

(a) the first arc of the curve diagram ends in a puncture or can be pulled tight so as to end in a puncture;

(b) the first arc cuts D_n into two disks, each of which contains at least two punctures.

For case (a) we notice that the first arc can be turned into the horizontal arc from -1 to the leftmost puncture, by an action of some appropriate element of B_n. There are now precisely M_{n-1} orbits of loose isotopy classes of curve diagrams of the remaining $n-2$ arcs in the $n-1$-punctured disk $D_n \backslash$(the first arc). So case (a) gives a contribution of M_{n-1} orbits.

The argument for case (b) is similar: the action of an appropriate element of B_n will turn the first arc of any curve diagram of type (b) into the vertical arc, oriented from bottom to top, having k punctures on its left and $n-k$ on its right, for some $k \in \{2, \ldots, n-2\}$. In this case, there should be $k-1$ arcs on the left and $n-k-1$ arcs on the right of the first arc, so there are $\binom{n-2}{k-1}$ ways to distribute the remaining $n-2$ arcs over the two sides. Finally, there are M_k respectively M_{n-k} orbits of loose isotopy classes of total curve diagrams on the disk on the left respectively on the right. □

6. Replacing finite type geodesics by curve diagrams

In this section we prove the main theorems on orderings of finite type. The strategy is to associate to every geodesic of finite type a curve diagram such that the (possibly partial) orderings arising from the geodesic and the curve diagram agree. Thus we obtain, via curve diagram orderings, a good understanding of finite type orderings.

Proof of Theorem 3.3 (a). If $D_n \backslash \gamma_\alpha$ has a path component which contains at least two holes, then we can push the boundary curve of this path component slightly into its interior, to make it disjoint from γ_α. A Dehn twist along such a curve will be a nontrivial element of B_n, and act trivially on γ_α. □

We now define *the curve diagram $C(\gamma_\alpha)$ associated to a geodesic γ_α* of finite type. It is a subset of γ_α, more precisely a union of segments of γ which start and end at self-intersection points. The diagram will be disjoint from the punctures, except that the last arc may fall into a puncture. For simplicity we shall write Γ for $C(\gamma_\alpha)$ and, as before, $\Gamma_{0\cup\cdots\cup i-1}$ for $\bigcup_{k=0}^{i-1} \Gamma_k$.