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IDEAL SOLUTIONS OF THE TARRY-ESCOTT PROBLEM OF DEGREE FOUR AND A RELATED DIOPHANTINE SYSTEM

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ABSTRACT. In this paper, the complete ideal symmetric solution in integers of the Tarry-Escott problem of degree four, that is, of the system of simultaneous equations $\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r$, $r = 1,2,3,4$, has been obtained. In addition, a parametric ideal non-symmetric solution has also been obtained, and this non-symmetric solution has been used to obtain a parametric solution of the diophantine system $\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r$, $r = 1,2,3,4$ and 6.

1. INTRODUCTION

The Tarry-Escott problem of degree k consists of finding two sets of integers $a_1, a_2, \dots, a_s$ and $b_1, b_2, \dots, b_s$ such that

$$\sum_{i=1}^{s} a_i^r = \sum_{i=1}^{s} b_i^r, \qquad r = 1,2,\dots,k. \tag{1}$$

There is a well-known theorem [6, p. 614] due to Frolov according to which the relations (1) imply that

$$\sum_{i=1}^{s} (Ma_i + K)^r = \sum_{i=1}^{s} (Mb_i + K)^r, \qquad r = 1,2,\dots,k, \tag{2}$$

where M and K are arbitrary integers. That is, if $(a_1, a_2, \dots, a_s; b_1, b_2, \dots, b_s)$ is a solution of the system (1), then

$$(Ma_1 + K, \dots, Ma_s + K; Mb_1 + K, \dots, Mb_s + K)$$

is also a solution of (1). This theorem is easily established by using the binomial theorem. If one solution of the system (1) is obtained from another through the application of this theorem, the two are called equivalent solutions. When we speak of distinct solutions, we refer to solutions that are not equivalent.

It follows from Frolov's theorem that for each solution of (1), there is an equivalent one such that $\sum_{i=1}^{s} a_i = 0 = \sum_{i=1}^{s} b_i$ and the greatest common divisor of all the integers $a_1, a_2, \ldots, a_s$ and $b_1, b_2, \ldots, b_s$ is 1, that is, $\gcd(a_i, b_i) = 1$. This is said to be the *reduced form* of the solution.

It is easily established [6, p. 616] that for a non-trivial solution of (1) to exist, we must have $s \geq (k+1)$. Solutions of the system of equations (1) are called ideal if $s = (k+1)$ and are of particular interest in several applications [6].

In order to reduce the number of equations of the system (1), the following simplifying conditions are often imposed:

$$a_i = -b_i\,, \quad i = 1, 2, \ldots, s\,, \quad \text{for } s \text{ odd,} \tag{3}$$

or

$$a_{s+1-i} = -a_i\,, \quad b_{s+1-i} = -b_i\,, \quad i = 1, 2, \ldots, s/2\,, \quad \text{for } s \text{ even.} \tag{4}$$

Solutions of (1) subject to the conditions (3) or (4) are called symmetric solutions. The conditions of symmetry, together with the condition $\gcd(a_i, b_i) = 1$, ensure that such solutions are in reduced form. Solutions of (1) obtained by the application of Frolov's theorem to a symmetric solution are also considered symmetric as they are equivalent to a symmetric solution. Solutions of (1) that are not symmetric are called non-symmetric.

The complete ideal solution (whether symmetric or non-symmetric) of the Tarry-Escott problem of degrees 2 and 3 has been given by Dickson [4, pp. 52, 55–58] but for higher degrees the complete ideal solution is not known. When $4 \leq k \leq 7$, parametric ideal solutions of (1) are known but these are all symmetric [2; 3, pp. 304–305; 5; 7, pp. 41–54]. However, these parametric solutions do not even give the complete ideal symmetric solution for any $k \geq 4$. Numerical ideal symmetric solutions of (1) have been given by Letac [8] for $k = 8$, by Letac as well as Smyth [10] for $k = 9$ and recently a numerical ideal symmetric solution for $k = 11$ was discovered through the combined efforts of Nuutti Kuosa, Jean-Charles Meyrignac and Chen Shuwen [9]. Parametric ideal non-symmetric solutions of (1) are not known for any $k \geq 4$. A numerical ideal non-symmetric solution has been given by Gloden [7, p. 25] when $k = 4$. Moreover, the aforementioned numerical ideal solutions for $k = 9$ given by Letac and Smyth provide non-symmetric ideal solutions for $k = 4$ [1, p. 10].

It is interesting to observe that ideal non-symmetric solutions of (1) can be used to generate solutions of the system of equations

$$\sum_{i=1}^{k+1} a_i^r = \sum_{i=1}^{k+1} b_i^r, \qquad r = 1, 2, \ldots, k, k+2. \tag{5}$$

This follows from a theorem given by Gloden [7, p. 24]. Symmetric ideal solutions cannot be used effectively for this purpose as the solutions obtained by applying this theorem hold trivially either for all odd or for all even values of r according as k is odd or even.

In this paper, we will obtain the complete ideal symmetric solution of the Tarry-Escott problem of degree four as well as a parametric ideal non-symmetric solution of this problem. We shall use the non-symmetric solution to obtain a parametric solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4, 6. \tag{6}$$

Parametric solutions of the system of equations (6) have not been obtained earlier.

2. The complete ideal symmetric solution of the Tarry-Escott problem of degree four

To obtain the complete ideal symmetric solution of degree four, we have to obtain a solution of the system of equations

$$\sum_{i=1}^{5} a_i^r = \sum_{i=1}^{5} b_i^r, \qquad r = 1, 2, 3, 4, \tag{7}$$

where $a_i = -b_i$, $i = 1, 2, \ldots, 5$. The four equations of the system (7) now reduce to the following two equations:

$$a_1 + a_2 + a_3 + a_4 + a_5 = 0 \tag{8}$$

and

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = 0\,. \tag{9}$$

Thus, to obtain the complete symmetric solution, in reduced form, of the diophantine system (7), we must obtain the complete solution in integers of the equations (8) and (9) such that $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$.

The equations (8) and (9) have trivial solutions in which one of the five integers is zero while the remaining four integers form two pairs, the sum of the integers in each pair being zero, as for example, $(x_1, x_2, -x_1, -x_2, 0)$.