§3. Main theorems

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THEOREM 1. *Let G be a locally compact commutative group, $\widehat{G}$ its dual group, the Haar measures on $G, \widehat{G}$ being determined as in §2.*

(i) *If $1 \leq p < 2$, then the contraction operator $\mathcal{F}_p$ given by (3) is surjective if and only if G is a finite group.*

(ii) *$\mathcal{F}\colon L^1(G) \to C_0(\widehat{G})$ is surjective if and only if G is a finite group; here $\mathcal{F}f = \widehat{f}$.*

Proof. (i) It is known that $\mathcal{F}_p$ is always injective; cf. [HR] vol. 2, (31.31), p. 231. If $\mathcal{F}_p$ is surjective, then $\mathcal{F}_p$ will be an isomorphism between $L^p(G)$ and $L^{p'}(\widehat{G})$; since $p' \neq p$ if $1 \leq p < 2$, this implies, according to the L^p-isomorphism theorem of §2, that $L^p(G)$ and hence $L^{p'}(\widehat{G})$ are finite dimensional i.e. G (and hence $\widehat{G}$) are finite groups. On the other hand, if G is a finite commutative group then it is a well-known elementary fact (see [HR] vol. 1, p. 357) that $\widehat{G}$ is isomorphic to G so that, for any p, q in $[1, \infty]$, $L^p(G)$ and $L^q(\widehat{G})$ are then of the same finite dimension equal to the order of the group G; hence, in particular, if G is a finite commutative group, $L^p(G)$ is isomorphic to $L^{p'}(\widehat{G})$ for $1 \leq p < 2$; the isomorphism can be realized via $\mathcal{F}_p$ since $\mathcal{F}_p$ is injective and $\dim L^p(G) = \dim L^{p'}(\widehat{G})$.

(ii) The proof here is perfectly similar; it uses the impossibility of an isomorphism between $L^1(\mu)$ and $C_0(Y)$ given in §2.

This completes the proof of Theorem 1.

The notations p', etc. are as in §2 for the following theorem as well; its proof uses the non-surjectivity given by Theorem 1 and an elementary inversion formula.

THEOREM 2. *Let G be an infinite commutative locally compact group and $2 < p < \infty$. Then no inequality of the form*

$$\|\widehat{f}\|_{p'} \leq M\|f\|_p \tag{5}$$

can hold for $f \in D$, D being a $L^p(G)$-dense linear subspace of $L^p(G) \cap L^1(G)$, whatever be the choice of M, $0 \leq M < \infty$.

Proof. We shall need the following simple facts:

(i) If $0 < a < c < b < \infty$ then, for any positive measure μ,

$$L^a(\mu) \cap L^b(\mu) \subset L^c(\mu).$$

This is evident from the following:

$$\int |f|^c d\mu = \int_{|f|\leq 1} |f|^c d\mu + \int_{|f|>1} |f|^c d\mu$$
$$\leq \int_{|f|\leq 1} |f|^a d\mu + \int_{|f|>1} |f|^b d\mu .$$

(ii) (Inversion formula for $L^2(G)$). If $f \in L^2(G)$ then

$$\tau \mathcal{F}_2(\mathcal{F}_2 f) = f$$

where $\tau g(x) = g(-x)$, $g\colon G \to \mathbf{C}$ being any function; cf. [HR] (31.17), p. 225.

(iii) If $\varphi \in L^a(G) \cap L^b(G)$, a, b being in $[1,2]$, then

$$\mathcal{F}_a \varphi = \mathcal{F}_b \varphi \qquad a.e.$$

This fact has already been explicitly mentioned in the introduction where an exact reference is given.

If (5) were to hold for $f \in D$, there would be a bounded linear operator T,

$$T\colon L^p(G) \to L^{p'}(\widehat{G})$$

such that

$$Tf = \widehat{f}, \qquad f \in D \subset L^p(G) \cap L^1(G).$$

Since $1 < p' < 2$, the Hausdorff-Young inequality gives a linear contraction S,

$$S\colon L^{p'}(\widehat{G}) \to L^p(G)$$

such that $S\varphi = \tau \mathcal{F}_{p'} \varphi$.

Now, if $f \in D$, f is in $L^2(G)$ (since $1 < 2 < p$; cf. (i) above) as well as in $L^1(G)$ (by hypothesis) so that

$$Tf = \widehat{f} \in L^2(\widehat{G}) \cap L^{p'}(\widehat{G}).$$

Thus, for $f \in D$,

$$S(Tf) = S(\mathcal{F}_2 f) = \tau \mathcal{F}_{p'}(\mathcal{F}_2 f) = \tau \mathcal{F}_2(\mathcal{F}_2 f) = f$$

by using the facts (ii) and (iii) given above. Since D is dense in $L^p(G)$ and the operator ST is continuous we deduce that

$$STf = f, \qquad f \in L^p(G),$$

which obviously implies that S must be surjective; this contradicts Theorem 1 thus establishing the impossibility of (5) for $f \in D$.

This completes the proof of Theorem 2.

REMARK. We observe that conversely, Theorem 1 can be deduced from Theorem 2; we shall not elaborate on this; however, our proof of Theorem 1 shows that its validity stems from a simple general result on L^p-spaces.

§4. HISTORICAL REMARKS

The inequality (1) was given first by Hausdorff [H] in 1923 for the groups $G = \mathbf{T}$ (with $\widehat{G} = \mathbf{Z}$) and $G = \mathbf{Z}$ (with $\widehat{G} = \mathbf{T}$). Hausdorff was inspired by the work of W. H. Young from 1912–13 who proved that the Fourier series of a function in L^p, $1 \leq p \leq 2$, had coefficients which were in $\ell^{p'}$ (and, in a suitable sense, vice-versa) for $p' = 2k$, a positive even integer, $p = 2k/(2k-1)$. Young did not formulate his results in terms of inequalities which were given first by Hausdorff (for all $p \in [1,2]$ and for the groups $G = \mathbf{T}$, $G = \mathbf{Z}$, i.e. for Fourier series). Hausdorff's proof, which is all but forgotten today, used Young's results for $p' = 2k$ and some of Young's techniques to carry out an interpolation argument for all the values of p, p', $1 \leq p \leq 2$, missing in Young's work. Hausdorff's paper [H] gives the exact references to W. H. Young's paper which were related to his work.

Shortly afterwards, after having heard of Hausdorff's inequalities, F. Riesz obtained independently (in [RF]) some Hausdorff-Young type inequalities, valid for series expansions in terms of arbitrary *bounded* orthogonal functions. This paper of F. Riesz was important not only because it showed that Hausdorff-Young type inequalities did not belong exclusively to the theory of Fourier series but also because F. Riesz (in collaboration with his colleague A. Haar) conjectured there the validity of a general "arithmetical" inequality for linear forms (in a finite number of variables) which they claimed to be enough for proving F. Riesz's theorem for orthogonal expansions.

It was this conjecture which seems to have led M. Riesz (F. Riesz's younger brother) to formulate and prove in 1927 ([RM]) his convexity theorem for bilinear forms and use it to deduce Hausdorff-Young-F. Riesz inequalities and many others. M. Riesz's work was exactly what A. Weil used in 1940 to establish (1) for general locally compact commutative groups in his book [W], p. 117. As is well-known, once the Plancherel theorem for a general $L^2(G)$, G locally compact commutative, is established (and this was done by Weil) the proof of (1) via M. Riesz's theorem is almost immediate. M. Riesz's work was simplified and much generalized by Thorin in 1938 (and later in 1948; exact references can be found in [DS] or in [HR]) which launched the later theory of interpolation of operators due to many well-known mathematicians which