4. The proof

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3. A Reduction

To prove the Spectral Mapping Theorem it suffices to verify that it holds for the polynomial ring $k[t_0, \dots, t_m]$ in variables $t_0, \dots, t_m$ over k, and for the polynomial $F(x) = t_0 + t_1 x + \dots + t_m x^m$. This is because any polynomial $G(x) = b_0 + b_1 x + \dots + b_m x^m$ with coefficients in a ring R containing k as a subring is the image of $F(x)$ by the map $g\colon k[t_0, \dots, t_m][x] \to R[x]$ defined by $g(a) = a$ for $a \in k$, by $g(t_i) = b_i$ for $i = 0, \dots, m$, and by $g(x) = x$. If we can prove the equality $\det F(M) = \prod_{i=1}^n F(\lambda_i)$ in $k[t_0, \dots, t_m]$ we obtain that $\det G(M) = g(\det F(M)) = \prod_{i=1}^n g(F(\lambda_i)) = \prod_{i=1}^n G(\lambda_i)$ in R.

4. The Proof

Clearly (2.1) holds when F is a constant a where it simply states that $\det(aI_n) = a^n$. We shall prove (2.1) for polynomials F of degree $m > 0$ by induction on m.

We first note that if $F(x)$ has a root λ in R, so that $F(x) = (x - \lambda)G(x)$ in $R[x]$, then (2.1) holds for $F(x)$. Indeed, $G(x)$ is of degree $m - 1$ so it follows from the induction hypothesis that $\det G(M) = \prod_{i=1}^n G(\lambda_i)$. Since $F(M) = (M - \lambda I_n)G(M)$ we obtain:

$$\begin{aligned}\det F(M) &= \det(M - \lambda I_n) \det G(M) \\ &= \prod_{i=1}^n (\lambda_i - \lambda) \prod_{i=1}^n G(\lambda_i) = \prod_{i=1}^n (\lambda_i - \lambda) G(\lambda_i) = \prod_{i=1}^n F(\lambda_i).\end{aligned}$$

As we saw in Section 3 it suffices to prove the result for the ring $Q = k[t_0, \dots, t_m]$ and the polynomial $F(x) = t_0 + t_1 x + \dots + t_m x^m$. Let x and y be independent variables over the ring Q. The polynomial $F(x) - F(y)$ in x with coefficients in $Q[y]$ has the root $x = y$. Hence, as we just observed, (2.1) holds for the polynomial $F(x) - F(y)$. We obtain the equation:

$$\det(F(M) - F(y)I_n) = \prod_{i=1}^n (F(\lambda_i) - F(y)) \tag{4.1}$$

in $Q[y]$.

The equation (2.1) is a consequence of (4.1). To see this we observe that $F(y)$ in $Q[y]$ is transcendent over Q, that is the element $F(y)$ in $Q[y]$ does not satisfy a polynomial relation $a_0 + a_1 F(y) + \dots + a_l F(y)^l = 0$ with coefficients a_i in Q and $a_l \neq 0$, because the coefficient $a_l t_m^l$ of the highest power y^{ml} of y that appears in the relation is non-zero. It follows that we can define a homomorphism of rings $h\colon Q[F(y)] \to Q$ by $h(a) = a$ for $a \in Q$, and $h(F(y)) = 0$. We apply the map h to both sides of (4.1) and obtain the equality (2.1).

5. Norms on algebras

The only properties of determinants that we used in the proof of the Spectral Mapping Theorem is that they are multiplicative, functorial and homogeneous. It is therefore natural to place the proof into the more general framework of norms on algebras. The advantage of this point of view is that we obtain a deeper understanding of the Spectral Mapping Theorem, and we obtain a natural connection with resultants of polynomials.

A *norm* N of *degree* n on a, not necessarily commutative, k-algebra A is a family of maps $N_R\colon R \otimes_k A \to R$, one for every commutative k-algebra R, that satisfies the conditions:

(1) $N_R(a \otimes 1) = a^n$ for all elements a in R.

(2) $N_R(uv) = N_R(u)N_R(v)$ for all elements u and v of $R \otimes_k A$.

(3) For every homomorphism $\varphi\colon R \to S$ of commutative k-algebras we have $\varphi N_R = N_S(\varphi \otimes \mathrm{id}_A)$.

A norm on an algebra may be described as a *multiplicative homogeneous polynomial law* (see Roby [R], or [B1], §9, Définition 3, p. 52).

For any map $B \to A$ of k-algebras the norm N on A *restricts* to a norm on B of degree n. Moreover, for every homomorphism of commutative rings $k \to k'$ the norm N on A *induces* a norm of degree n on the k'-algebra $k' \otimes_k A$.

Let N be a norm of degree n on a k-algebra A. Denote by $k[t]$ the k-algebra of polynomials in the variable t with coefficients in k. For every element α in A the polynomial in $k[t]$:

$$P_\alpha(t) = P_\alpha^N(t) = N_{k[t]}(t - \alpha)$$

is called the *characteristic polynomial* of α. The *trace* $\mathrm{Tr}^N(\alpha)$ of α is the element in k such that $-\mathrm{Tr}^N(\alpha)$ is the coefficient of t^{n-1} in $P_\alpha(t)$.

We note that $P_\alpha(0) = (-1)^n N_k(\alpha)$.