6. NORMS AND RESULTANTS

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5.1. LEMMA. *Let N be a norm of degree n on a k-algebra A. For each element α of A the characteristic polynomial $P_\alpha^N(t) = N_{k[t]}(t-\alpha)$ is monic of degree n.*

Moreover, the trace Tr^N *is a k-linear map $A \to k$.*

Proof. Let s, t, u, v be independent variables over the ring k. For each element β in A the norm $N_{k[s,t,u]}(t - \alpha s - \beta u)$ is a polynomial in $k[s,t,u]$. Since N is of degree n we have that $N_{k[s,t,u,v]}(vt - \alpha vs - \beta vu) = v^n N_{k[,s,t,u]}(t - \alpha s - \beta u)$. It follows that $N_{k[s,t,u]}(t - \alpha s - \beta u)$ is homogeneous of degree n in $k[s,t,u]$. In particular the coefficient of t^{n-1} is of the form $as+bu$ with a and b in k. By evaluating the polynomial $N_{k[s,t,u]}(t - \alpha s - \beta u)$ at $s=0$, $u=0$, it follows that the coefficient to t^n is equal to 1. Hence $N_{k[t]}(t-\alpha)$ is a monic polynomial of degree n, and $a = -\mathrm{Tr}^N(\alpha)$. Similarly, $b = -\mathrm{Tr}^N(\beta)$. Hence we have that $\mathrm{Tr}^N(\alpha s + \beta u) = -(as+bu) = \mathrm{Tr}^N(\alpha)s + \mathrm{Tr}^N(\beta)t$. Specializing s and t to any pair of elements of k the second part of the Lemma follows. □

5.2. EXAMPLE. Let M be a free module of rank n over k, or more generally a projective k-module of constant rank n. Then the determinant defines a norm of degree n on $\mathrm{End}_k(M)$.

Let A be a k-algebra which is free of rank n as a k-module. Left multiplication by elements of A define an injection $A \to \mathrm{End}_k(A)$ of k-algebras. By restriction we obtain a norm of degree n on A.

6. NORMS AND RESULTANTS

Let $F(x) = f_0 + \cdots + f_m x^m$ and $P(x) = p_0 + \cdots + p_n x^n$ be polynomials of degree m, respectively n in the k-algebra $k[x]$ of polynomials in the variable x with coefficients in k. The *resultant* $\mathrm{Res}(F,P)$ of F and P is the determinant of the $(m+n)\times(m+n)$-matrix $D(F,P)$ whose columns are the coefficients of the polynomials $F, xF, \dots, x^{n-1}F, P, xP, \dots, x^{m-1}P$. Note that the definition is asymmetric in F and P in the sense that $\mathrm{Res}(F,P) = (-1)^{mn}\mathrm{Res}(P,F)$.

When P is monic the resultant is equal to the determinant of the endomorphism induced by multiplication by F on the free k-module $k[x]/(P(x))$ of rank n. To see this we note that for $i = 0, \dots, n-1$ we can write $x^i F = Q_i P + R_i$ in $k[x]$, where $Q_i(x)$ and $R_i(x)$ are of degrees at most $m-1$, respectively $n-1$. It follows that the determinant of $D(F,P)$ is equal to the determinant of the $(m+n)\times(m+n)$-matrix $B(F,P)$ whose columns are the coefficients of the polynomials $R_0, \dots, R_{n-1}, P, xP, \dots, x^{m-1}P$. We see that the $n \times n$-block $C(F,P)$ in the upper left corner of $B(F,P)$ is the matrix $C(F,P)$ of the map induced by multiplication by F on $k[x]/(P(x))$, and the $m \times m$-block in the lower right corner is upper triangular with 1's on the diagonal. Moreover, the entries of $C(F,P)$ are the only non-zero entries in the first n columns of $B(F,P)$. It follows that $\operatorname{Res}(F,P) = \det C(F,P)$, as we claimed.

6.1. EXAMPLE. When P is a monic polynomial we saw in Example 5.2 that the k-algebra $k[x]/(P(x))$ which is free of rank n as a k-module has a canonical norm. Via the canonical map $k[x] \to k[x]/(P(x))$ we obtain a canonical norm N'_P on $k[x]$. The above interpretation of the resultant can then be written as

$$(N'_P)_R(F) = \operatorname{Res}(F,P) \tag{6.1.1}$$

for all commutative k-algebras R and all polynomials $F(x)$ in $R[x] = R \otimes_k k[x]$. By an easy computation of the determinant defining $\operatorname{Res}(t-x, P)$, we obtain that the characteristic polynomial of x with respect to N'_P is

$$P_x^{N'_P}(t) = P(t).$$

6.2. EXAMPLE. We shall introduce a second important norm on $k[x]$. Let $P(x)$ be a monic polynomial of degree n in the k-algebra $k[x]$. There is a canonical ring extension $k \subseteq k' = k[\lambda_1, \dots, \lambda_n]$ such that $P(x)$ splits as $P(x) = \prod_{i=1}^n (x - \lambda_i)$ in $k'[x]$. The extension is obtained by induction starting with $k = k_0$ and $P_0(x) = P(x)$, and constructing $k_i = k[\lambda_1, \dots, \lambda_i]$ and $P_i(x) \in k[\lambda_1, \dots, \lambda_i][x]$ from k_{i-1} and P_{i-1}, for $i = 1, 2, \dots, n$, by $k_i = k_{i-1}[x]/(P_{i-1}(x)) = k_{i-1}[\lambda_i]$, where λ_i is the class of x, and by $P_i(x) = P_{i-1}(x)/(x - \lambda_i)$. We note that k' is a free k-module of rank $n!$. The algebra k' is sometimes called the *universal decomposition algebra for* P (see [B1], §6, p. 68).

For every commutative k-algebra R and every polynomial G in $R[x] = R \otimes_k k[x]$ we have that $\prod_{i=1}^n G(\lambda_i)$ is symmetric in $\lambda_1, \dots, \lambda_n$, and consequently lies in the image of the inclusion $R \subseteq k' \otimes_k R$. We obtain a map $(N''_P)_R \colon R \otimes_k k[x] \to R$ defined by $(N''_P)_R(G) = \prod_{i=1}^n G(\lambda_i)$. In this way we obtain a norm N''_P of degree n on $k[x]$ and the characteristic polynomial of x with respect to the norm N''_P is

$$P_x^{N''_P}(t) = \prod_{i=1}^{n} (t - \lambda_i) = P(t).$$