3. The Witt group of polynomial rings

COROLLARY 2.4. *Suppose that A is a ring with involution, in which 2 is invertible. Then*

$$H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = H^2(\mathbf{Z}/2, K_{-1}(A)).$$

3. THE WITT GROUP OF POLYNOMIAL RINGS

THEOREM 3.1. *Let A be an associative ring with involution, in which 2 is invertible. Let ϵ be 1 or -1 and let W be the Witt group functor of ϵ-hermitian spaces. The natural homomorphism*

$$W(A) \longrightarrow W(A[t])$$

is an isomorphism.

Proof. It suffices to show that the homomorphism $W(A[t]) \to W(A)$ given by the evaluation at $t = 0$ is an isomorphism. Surjectivity is obvious. To prove injectivity let (P, α) be a space over $A[t]$ and $(P(0), \alpha(0))$ its reduction modulo t. Suppose that $(P(0), \alpha(0))$ is isometric to some hyperbolic space $H(Q)$. Choosing a projective module Q' such that $Q \oplus Q'$ is free and adding to (P, α) the space $H(Q'[t])$ we may assume that $P(0)$ is the hyperbolic space over a free module. The class of P in $K_0(A[t])/K_0(A) = N_+(A)$ is a symmetric element. By Corollary 2.4 it can be written as $a + a^*$, hence, adding to (P, α) a suitable free hyperbolic space, we may assume that (P, α) is of the form

$$H(A^n[t]) \perp (R \oplus R^*, \beta).$$

Let R' be an $A[t]$-module such that $R \oplus R'$ is free. Adding to (P, α) the hyperbolic space $H(R')$ we are reduced to the case in which P is free and α is an invertible ϵ-hermitian matrix with entries in $A[t]$.

LEMMA 3.2. *Let $\alpha = \epsilon\alpha^* \in \mathrm{M}_n(A[t])$ be any ϵ-hermitian matrix. There exist an integer m and a matrix $\tau \in \mathrm{GL}_{n+2m}(A[t])$ (actually in $\mathrm{E}_{n+2m}(A[t])$) such that*

$$\tau^* \begin{pmatrix} \alpha & 0 \\ 0 & \chi \end{pmatrix} \tau = \alpha_0 + t\alpha_1 ,$$

where α_0 and α_1 are constant matrices and χ is a sum of hyperbolic blocks $\begin{pmatrix} 0 & 1 \\ \epsilon 1 & 0 \end{pmatrix}$ of various sizes.

Proof of the lemma. Write $\alpha = \gamma + \delta t^N$, where δ is constant and γ of degree less than N. Assume that N is at least 2. Since δ is ϵ-hermitian and 2 is invertible in A we can write $\delta = \sigma + \epsilon\sigma^*$. Then

$$\begin{pmatrix} 1 & t & -\sigma^* t^{N-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma + \sigma t^N + \epsilon\sigma^* t^N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\sigma t^{N-1} & 0 & 1 \end{pmatrix}$$

is of degree $\leq N-1$ and after $N-1$ such transformations we get a linear matrix. □

Writing $\alpha = \alpha_0 + t\alpha_1$ as $\alpha_0(1+\nu t)$ we see immediately that, α being invertible, ν is nilpotent. The formal power series

$$\tau = (1+\nu t)^{-1/2} = \sum \binom{-1/2}{k} (\nu t)^k$$

is a polynomial. From $\alpha = \epsilon\alpha^*$ we get $\alpha_0^* = \epsilon\alpha_0$ and $\nu^*\alpha_0^* = \epsilon\alpha_0\nu$. This implies that $\tau^*\alpha_0^* = \epsilon\alpha_0\tau$ and therefore

$$\tau^*\alpha\tau = \tau^*\alpha_0(1+\nu t)\tau = \alpha_0\tau(1+\nu t)\tau = \alpha_0 .$$

This proves that (P, α) is Witt equivalent to $(P(0), \alpha(0))$ and is, therefore, hyperbolic. □

4. The Witt group of torsion modules

Let M be a finitely generated right $A[t]$-module and suppose that it is a t-torsion module and that it is projective as an A-module. Obviously, it will be finitely generated over A. We denote by $M^\sharp$ the left $A[t]$-module $\mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t])$ and we consider it as a right module through the involution on $A[t]$.

Recall that, as an A-module, the quotient $A[t,t^{-1}]/A[t]$ can be written as a direct sum

$$A[t,t^{-1}]/A[t] = At^{-1} \oplus At^{-2} \oplus \cdots .$$

Thus, to any $f \in \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t])$ we can associate an A-linear map $f_{-1}\colon M \to A$, which is defined as the composite of f with the projection onto At^{-1}.