6. The residue

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6. The residue

In this section we construct a residue map

$$Res\colon W'(A[t,t^{-1}]) \to W(A)$$

satisfying R_1 and R_2 of §5.

The definition of Res will be preceded by a few preliminaries.

LEMMA 6.1. *Let P_0 be a (finitely generated) projective A-module and define $M(\alpha)$ by the exact sequence*

$$0 \longrightarrow P_0[t] \xrightarrow{\alpha} P_0^*[t] \longrightarrow M(\alpha) \longrightarrow 0,$$

where α is $A[t]$-linear. Suppose that its localization $\alpha_t\colon P_0[t,t^{-1}] \to P_0[t,t^{-1}]$ is an isomorphism. Then, as an A-module, $M(\alpha)$ is finitely generated and projective.

Proof. Decompose $P_0[t,t^{-1}]$ as a direct sum $P_0[t] \oplus t^{-1}P_0[t^{-1}]$ of A-modules. Let π be the projection onto the first summand. Then $\beta = \pi \circ \alpha_t^{-1}|_{P_0^*[t]}$ is an A-linear splitting of α. Hence $M(\alpha)$ is A-projective. It is also finitely generated as an $A[t]$-module, hence, being annihilated by a power of t, it is finitely generated as an A-module. □

Let $M = M(\alpha)$ be as in the previous lemma. Assume that α is ϵ-symmetric. We define a pairing

$$M \times M \to A[t,t^{-1}]/A[t]$$

by $\langle \bar{a}, \bar{b} \rangle = a(\alpha_t^{-1}(b))$, where a and b are representatives in $P_0^*[t]$ of $\bar{a}, \bar{b} \in M$.

LEMMA 6.2. *If α is ϵ-hermitian, then $\langle , \rangle$ is a perfect ϵ-hermitian pairing.*

Proof. Since α_t is ϵ-hermitian, denoting by $x \mapsto x^\circ$ the involution on A we have

$$\langle \bar{a}, \bar{b} \rangle = a(\alpha_t^{-1}(b)) = \epsilon(b(\alpha_t^{-1}(a)))^\circ = \epsilon\langle \bar{b}, \bar{a} \rangle^\circ .$$

This proves the first assertion.

We now check that the adjoint of $\langle , \rangle$

$$\chi\colon M \to \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]),$$

defined as $\chi(\bar{a})(\bar{b}) = \langle \bar{a}, \bar{b} \rangle$, is an isomorphism. We first prove injectivity. Suppose that, for some a and every x in M, $\chi(\bar{a})(\bar{x}) = 0$. This means that $a(\alpha_t^{-1}(x)) \in A[t]$ for every $x \in P_0^*[t]$. We only have to show that $\alpha_t^{-1}(a) \in P_0[t]$. Consider the diagram

$$\begin{array}{ccc} P_0[t] & \xrightarrow{\sim} & \mathrm{Hom}_{A[t]}(P_0^*[t], A[t]) \\ \downarrow & & \downarrow \\ P_0[t, t^{-1}] & \xrightarrow{\sim} & \mathrm{Hom}_{A[t]}(P_0^*[t], A[t, t^{-1}]) \end{array}$$

where the horizontal arrows are the canonical ones. Since $P_0[t]$ is projective (and finitely generated!) over $A[t]$, they both are isomorphisms. Therefore an element $b \in P_0[t, t^{-1}]$ is in $P_0[t]$ if and only if, for any $x \in P_0^*[t]$, $x(b)$ is in $A[t]$. This is indeed the case for $b = \alpha_t^{-1}(a)$ because $x(\alpha_t^{-1}(a)) = \epsilon(a(\alpha_t^{-1}(x)))^\circ \in A[t]$ by the very assumption on a. Thus injectivity is proved. We now check that χ is surjective. Let $\bar{f}\colon M \to A[t, t^{-1}]/A[t]$ be an $A[t]$-linear map. Since $P_0[t]^*$ is projective, there exits an f which makes the right hand square of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0[t] & \xrightarrow{\alpha} & P_0[t]^* & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \bar{f}} & & \\ 0 & \longrightarrow & A[t] & \longrightarrow & A[t, t^{-1}] & \xrightarrow{q} & A[t, t^{-1}]/A[t] & \longrightarrow & 0 \end{array}$$

commute, p and q being the canonical surjections. Clearly $q \circ f \circ \alpha = 0$, hence there exists an $A[t]$-linear map $a\colon P_0[t] \to A[t]$ such $f \circ \alpha = i \circ a$, i being the inclusion $A[t] \to A[t, t^{-1}]$. We claim that $\chi(a) = \bar{f}$. For this it suffices to show that for any $b \in P_0[t]^*$ we have $a(\alpha_t^{-1}(b)) \equiv f(b)$ modulo $A[t]$. We denote by a_t the localization of a at t and by $f_t\colon P_0[t, t^{-1}]^* \to A[t, t^{-1}]$ the unique $A[t, t^{-1}]$-linear extension of f. Observing that $\alpha_t^{-1}(a) = a_t \circ \alpha_t^{-1}$ we get the following relations:

$$a(\alpha_t^{-1}(b)) = (a_t \circ \alpha_t^{-1})(b) = f_t(b) = f(b).$$

This proves that χ is surjective. $\square$

Let now $(P_0[t, t^{-1}], \alpha)$ be an ϵ-hermitian space. For any natural integer n for which $t^{2n}\alpha(P_0[t]) \subseteq P_0[t]^*$ we define $M(\alpha, n)$ by the exact sequence

$$0 \longrightarrow P_0[t] \xrightarrow{t^{2n}\alpha} P_0^*[t] \longrightarrow M(\alpha, n) \longrightarrow 0$$

and equip it with the ϵ-hermitian structure defined above:

$$\langle \bar{a}, \bar{b} \rangle = a((t^{2n}\alpha_t)^{-1}(b)).$$

LEMMA 6.3. *Let $\psi\colon (P_0[t,t^{-1}],\alpha) \to (Q_0[t,t^{-1}],\beta)$ be an isometry and assume that $\psi(P_0[t]) \subseteq Q_0[t]$, $\alpha(P_0[t]) \subseteq P_0[t]^*$ and $\beta(Q_0[t]) \subseteq Q_0[t]^*$. Then $M(\alpha)$ and $M(\beta)$ are Witt equivalent t-torsion spaces.*

Proof. Consider the diagram

$$\begin{tikzcd}
 & & 0 & & \\
 & 0 \arrow[d] & K \arrow[u] & & \\
0 \arrow[r] & P_0[t] \arrow[r, "\alpha"] \arrow[d, "\psi"] & P_0[t]^* \arrow[r, "q_\alpha"] \arrow[u, "\hat{q}"] & M(\alpha) \arrow[r] & 0 \\
0 \arrow[r] & Q_0[t] \arrow[r, "\beta"] \arrow[d, "q"] & Q_0[t]^* \arrow[r, "q_\beta"] \arrow[u, "\psi^*"] & M(\beta) \arrow[r] & 0 \\
 & L \arrow[d] & 0 \arrow[u] & & \\
 & 0 & & &
\end{tikzcd}$$

By Lemma 6.1 the module L, viewed as an A-module, is finitely generated and projective. The map ψ^* is obtained from the map ψ by dualizing over $A[t]$. We denote the cokernel of ψ^* by K and we denote the canonical map $P_0[t]^* \to K$ by $\hat{q}$. One may observe that K is isomorphic to $L^\sharp$ (see §4 for the notation) but we will not use this observation.

The $A[t]$-linear map $\theta = q_\alpha \circ \psi^*\colon Q_0[t]^* \to M(\alpha)$ induces a map $\overline{\theta}\colon M(\beta) \to \theta(Q_0[t]^*)/\theta(\beta(Q_0[t]))$. The statement will be deduced from the following claims.

(1) The map $\overline{\theta}$ is an $A[t]$-linear isomorphism.

(2) The map $\hat{q}$ induces an $A[t]$-linear isomorphism

$$\rho\colon M(\alpha)/\theta(Q_0[t]^*) \to K\,.$$

(3) $\theta(\beta(Q_0[t]))$ is a sublagrangian of $M(\alpha)$.

(4) $(\theta(\beta(Q_0[t]))^\perp = \theta(Q_0[t]^*)$.

(5) The map $\overline{\theta}$ is an isometry of t-torsion spaces.

In fact, by (4), (5) and Theorem 4.5, $M(\beta)$ is Witt equivalent to $M(\alpha)$.

We now prove the claims. The surjectivity of $\overline{\theta}$ is clear. To show injectivity, suppose that $x \in \ker(\theta)$. Choose a lift $\widetilde{x} \in Q_0[t]^*$ of x. There exist a $y \in Q_0[t]$ and a $z \in P_0[t]$ such that $\psi^*(\beta(y)-\widetilde{x}) = \alpha(z)$. Replacing α by $\psi^* \circ \beta \circ \psi$ we get $\psi^*(\widetilde{x}) = \psi^*(\beta(y-\psi(z)))$. Since ψ^* is injective, this shows that $\widetilde{x} \in \mathrm{Im}(\beta)$ and hence $x = 0$.

To prove (2) observe that, since $\hat{q} \circ \alpha = \hat{q} \circ \psi^* \circ \beta \circ \psi = 0$, $\hat{q}$ induces a surjective map $\rho\colon M(\alpha)/\theta(Q_0[t]^*) \to K$. Injectivity is also clear.

To prove (3) we first observe that $\theta(\beta(Q_0[t]))$ is a direct factor (as an A-module) of $M(\alpha)$. In fact, by (2), $\theta(Q_0[t]^*)$ is a direct factor (as an A-module) of $M(\alpha)$ and, by (1), $\theta(\beta(Q_0[t]))$ is a direct factor of $\theta(Q_0[t]^*)$. For any two elements $a, b \in P_0[t]^*$ let us denote by $\langle a,b\rangle_\alpha$ the element $a(\alpha_t^{-1}(b))$, and similarly for $\langle a,b\rangle_\beta$. We then have

$$\langle a,b\rangle_\beta = \langle \psi^*(a), \psi^*(b)\rangle_\alpha$$

because ψ_t is an isometry. Let now $\overline{a}, \overline{b} \in \theta(\beta(Q_0[t]))$ and $x, y \in Q_0[t]$ such that $a = \psi^*(\beta(x))$ and $b = \psi^*(\beta(y))$ are preimages of a and b. We have to check that $\langle \overline{a}, \overline{b}\rangle = 0$. This is the same as saying that $\langle a,b\rangle_\alpha$ is in $A[t]$. This is indeed the case because

$$\langle a,b\rangle_\alpha = \langle \psi^*(\beta(x)), \psi^*(\beta(y))\rangle_\alpha = \langle \beta(x), \beta(y)\rangle_\beta = \beta(x)(y) \in A[t]\,.$$

We now prove (4). For any $\overline{a} \in \theta(\beta(Q_0[t]))$ and any $\overline{b} \in M(\alpha)$ we choose preimages a and b of the form $a = \psi^*(\beta(x))$ and $b = \psi_t^*(y)$ with $x \in Q_0[t]$ and $y \in Q_0[t,t^{-1}]^*$. Then we have

$$\langle a,b\rangle_\alpha = \langle \psi^*(\beta(x)), \psi_t^*(y)\rangle_\alpha = \langle \beta(x), y\rangle_\beta = \epsilon \cdot y(x)^\circ\,,$$

which shows that, for any $y \in Q_0[t,t^{-1}]^*$, $\langle \psi^*(\beta(Q_0[t])), b\rangle_\alpha$ is in $A[t]$ if and only if $y \in Q_0[t]^*$, which is equivalent to $\overline{b} \in \theta(Q_0[t]^*)$.

We now prove (5). We already know that $\overline{\theta}$ is an $A[t]$-linear isomorphism. A computation like the one above proves that it is an isometry. □

COROLLARY 6.4. *Let $(P_0[t,t^{-1}], \alpha)$ be an ϵ-hermitian space. Let n be such that $t^{2n}\alpha(P_0[t]) \subseteq P_0[t]^*$. Then the class of $M(\alpha,n)$ in $W_{tors}(A[t])$ does not depend on the choice of n.*

COROLLARY 6.5. *Let $(P_0[t,t^{-1}], \alpha)$ and $(P_0[t,t^{-1}], \beta)$ be isometric spaces and assume that for some natural integers m and n, $t^{2m}\alpha(P_0[t]) \subseteq P_0[t]^*$ and $t^{2n}\beta(P_0[t]) \subseteq P_0[t]^*$. Then $M(\alpha,m)$ and $M(\beta,n)$ are Witt equivalent t-torsion spaces.*

Proof. Let $\psi\colon (P_0[t,t^{-1}], t^{2m}\alpha) \to (P_0[t,t^{-1}], t^{2n}\beta)$ be an isometry and let k be a natural integer such that $t^k\psi(P_0[t]) \subseteq P_0[t]^*$. Then $t^k\psi\colon (P_0[t,t^{-1}], t^{2m}\alpha) \to (P_0[t,t^{-1}], t^{2n+2k}\beta)$ is an isometry and, by Lemma 6.3, $M(\alpha,m)$ and $M(\beta,n+k)$ are Witt equivalent. Hence, by Corollary 6.4, $M(\alpha,m)$ and $M(\beta,n)$ are Witt equivalent as well. □

PROPOSITION 6.6. *Associating to any space $(P_0[t,t^{-1}],\alpha)$ the torsion space $M(\alpha,n)$ (for a suitable n) yields a homomorphism*

$$res\colon W'(A[t,t^{-1}]) \to W_{tors}(A[t]).$$

Proof. By Corollary 6.5, associating to the isometry class of a space $(P_0[t,t^{-1}],\alpha)$ the Witt class of the t-torsion space $M(\alpha,n)$ for some suitable n is a well defined map. It is obvious that the orthogonal sum of two spaces is mapped to the corresponding sum of t-torsion spaces, hence this map induces a homomorphism $\omega\colon K_H \to W_{tors}(A[t])$, where K_H is the Grothendieck group of ϵ-hermitian spaces of the form $(P_0[t,t^{-1}],\alpha)$. It is clear from the definition of $M(\alpha,n)$ that a standard hyperbolic space $H(Q_0[t,t^{-1}])$ is mapped to zero, hence ω induces a homomorphism $res\colon W'(A[t,t^{-1}]) \to W_{tors}(A[t])$. □

If we compose res with $\partial^W\colon W_{tors}(A[t]) \to W(A)$ we get a homomorphism

$$Res = \partial^W \circ res\colon W'(A[t,t^{-1}]) \to W(A)$$

which we call *residue.*

THEOREM 6.7. *The residue*

$$Res\colon W'(A[t,t^{-1}]) \to W(A)$$

satisfies the following two properties:

R_1 : *For any constant space $\xi \in W(A) \subset W(A[t,t^{-1}])$, $Res(\xi) = 0$.*

R_2 : *For any constant space $\xi \in W(A)$, $Res(t\cdot\xi) = \xi$.*

Proof. The two properties immediately follow from the construction of res. □

An amusing application of the existence of Res is the following result.

PROPOSITION 6.8. *Let A be a commutative semilocal ring in which 2 is invertible. Let (P,α) be a quadratic space over A. If (P,α) is isometric to $(P, t\cdot\alpha)$ over $A[t,t^{-1}]$, then (P,α) is hyperbolic.*

Proof. Let ξ be the class of (P,α) in $W(A)$. In $W'(A[t])$ we have $\xi = t\cdot\xi$. Applying *Res* to both sides we obtain $\xi = 0$. Since A is semilocal, by Witt's cancelletion theorem we conclude that (P,α) is hyperbolic. $\square$

7. THE WITT GROUP OF LAURENT POLYNOMIALS

Let $W'(A[t,t^{-1}])$ be the group defined in the introduction.

THEOREM 7.1. *Let A be an associative ring with involution in which 2 is invertible. Let*
$$\varphi\colon W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$$
be the canonical homomorphism.

(a) *If $H^2(\mathbf{Z}/2, K_{-1}(A)) = 0$, then φ is surjective.*

(b) *If $K_0(A) = K_0(A[t]) = K_0(A[t,t^{-1}])$, then φ is an isomorphism.*

Proof of (a). Corollary 2.4 implies that
$$H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = 0\,.$$
This means that every projective $A[t,t^{-1}]$-module P is in the same class as some projective module of the form
$$P_0[t,t^{-1}] \oplus Q \oplus Q^*\,,$$
where P_0 is a projective A-module. Therefore, adding to a space (P,α) a hyperbolic space $H(Q')$ with $Q\oplus Q'$ free, we may assume that P is of the form $P_0[t,t^{-1}]$. This means precisely that the class of (P,α) is in the image of $W'(A[t,t^{-1}])$. $\square$

Proof of (b). Surjectivity is obvious, because by assumption every projective $A[t,t^{-1}]$-module is stably extended from A. Suppose that the class of a space $(P_0[t,t^{-1}],\alpha)$ vanishes in $W(A[t,t^{-1}])$. This means that, for some Q and R, there exists an isometry
$$(P_0[t,t^{-1}],\alpha) \perp H(Q) \simeq H(R)\,.$$
Adding to both sides a suitable $H(A[t,t^{-1}]^n)$ we may replace Q and R by extended modules $Q_0[t,t^{-1}]$ and $R_0[t,t^{-1}]$. Then the isometry means precisely that the class of $(P_0[t,t^{-1}],\alpha)$ vanishes in $W'(A[t,t^{-1}])$. $\square$