8. TWO COUNTEREXAMPLES

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We can restate assertion (b) of Theorem 7.1 as follows.

THEOREM 7.2. *Let A be an associative ring with involution, in which 2 is invertible. Assume that $K_0(A) = K_0(A[t]) = K_0(A[t,t^{-1}])$. Then there exists a natural homomorphism Res such that the sequence*

$$0 \longrightarrow W(A) \longrightarrow W(A[t,t^{-1}]) \xrightarrow{Res} W(A) \longrightarrow 0$$

is split exact. The homomorphism Res restricts to an isomorphism of $t \cdot W(A)$ onto $W(A)$.

8. TWO COUNTEREXAMPLES

In this section we show that the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$, in general, is neither surjective nor injective.

EXAMPLE 8.1. We first recall the Mayer-Vietoris sequence associated to a cartesian square of commutative rings (see [1], Ch. IX, Corollary 5.12). Let

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f"'] & S \arrow[d, "g"] \\ \bar{R} \arrow[r] & \bar{S} \end{tikzcd}$$

be a cartesian diagram of commutative rings, with f or g surjective. Denote by $\widetilde{K_0}$ the kernel of the rank function on K_0. Then there is a commutative diagram with exact rows

$$\begin{tikzcd} K_1(\bar{R}) \times K_1(S) \arrow[r] \arrow[d, "\det"] & K_1(\bar{S}) \arrow[r] \arrow[d, "\det"] & \widetilde{K_0}(R) \arrow[r] \arrow[d, "\wedge^{\max}"] & \widetilde{K_0}(\bar{R}) \times \widetilde{K_0}(S) \arrow[r] \arrow[d, "\wedge^{\max}"] & \widetilde{K_0}(\bar{S}) \arrow[d, "\wedge^{\max}"] \\ \mathbf{G}_m(\bar{R}) \times \mathbf{G}_m(S) \arrow[r] & \mathbf{G}_m(\bar{S}) \arrow[r] & \mathrm{Pic}(R) \arrow[r] & \mathrm{Pic}(\bar{R}) \times \mathrm{Pic}(S) \arrow[r] & \mathrm{Pic}(\bar{S}) \end{tikzcd}$$

Let A be the local ring at the origin of the complex plane curve $Y^2 = X^2 - X^3$, $\widetilde{A}$ the normalisation of A and $\mathfrak{c}$ the conductor of $\widetilde{A}$ in A. Applying the big diagram above to the cartesian squares

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \widetilde{A} \arrow[d] \\ (A/\mathfrak{c}) \arrow[r] & (\widetilde{A}/\mathfrak{c}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A[t,t^{-1}] \arrow[r] \arrow[d] & \widetilde{A}[t,t^{-1}] \arrow[d] \\ (A/\mathfrak{c})[t,t^{-1}] \arrow[r] & (\widetilde{A}/\mathfrak{c})[t,t^{-1}] \end{tikzcd}$$

it is easy to see that $\widetilde{K_0}(A[t,t^{-1}]) = \mathbf{C}^* \oplus \mathbf{Z} = \mathrm{Pic}(A[t,t^{-1}])$. This shows that a projective $A[t,t^{-1}]$-module P is stably free if and only if its maximal exterior power $\bigwedge^{\max}(P)$ is isomorphic to $A[t,t^{-1}]$.

Let I be an ideal representing $(1,1)$ in $\mathbf{C}^*\oplus\mathbf{Z} = \mathrm{Pic}(A[t,t^{-1}])$. The module underlying the space $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ is free. In fact it is stably free because its determinant is trivial, hence, by a well-known cancellation theorem it is free. This shows that $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ is a quadratic space of the form $(P_0[t,t^{-1}],\alpha)$ with P_0 free of rank 6 over A. Clearly this space represents the zero element of $W(A[t,t^{-1}])$. We claim that its class in $W'(A[t,t^{-1}])$ is not trivial.

Since A is local, projective modules extended from A are free. If $H(I\oplus A[t,t^{-1}]\oplus A[t,t^{-1}])$ were hyperbolic in $W'(A[t,t^{-1}])$ it would be stably isometric to $H(A[t,t^{-1}] \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ and hence, by the quadratic cancellation theorem (see [4], VI, 6.2.5), it would be isometric to it. Recall that, for any commutative ring R in which 2 is invertible and any finitely generated projective R-module P, the even Clifford algebra C_0 of $H(P)$ is of the form

$$C_0 = \mathrm{End}_R(\textstyle\bigwedge^{even}(P)) \times \mathrm{End}_R(\textstyle\bigwedge^{odd}(P)),$$

where $\bigwedge^{even}(P)$ (respectively $\bigwedge^{odd}(P)$) is the even (respectively odd) part of the exterior algebra of P. In the case $P = I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}]$ we have

$$C_0 = \mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2) \times \mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2).$$

Suppose now that $H(I \oplus A[t,t^{-1}]^2)$ and $H(A[t,t^{-1}]^3)$ are isometric. In this case their even Clifford algebras would be isomorphic, hence the algebra $\mathrm{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2)$ would be a 4×4 matrix algebra. By Morita theory the module $A[t,t^{-1}]^2 \oplus I^2$ would be of the form J^4 for some invertible ideal J. Taking the fourth exterior power of both sides we would have $I^2 = J^4$, which is impossible because I represents $(1,1)$ in $\mathbf{C}^* \oplus \mathbf{Z}$.

This shows that, even for a one-dimensional local domain, the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ may fail to be injective.

EXAMPLE 8.2. We define a commutative ring A by the cartesian diagram of real algebras

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{R}[X,Y] \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{R} & \xrightarrow{\iota} & C \end{array} \tag{1}$$

where $C = \mathbf{R}[x, y] = \mathbf{R}[X, Y]/(X^2 + Y^2 - 1)$, π is the canonical projection and ι the canonical injection. Then $C \oplus C$ is the direct sum of its two submodules

$$P = C\tfrac{1}{2}(y+1, -x) + C\tfrac{1}{2}(-x, 1-y) \quad \text{and} \quad P' = C\tfrac{1}{2}(1-y, x) + C\tfrac{1}{2}(x, 1+y)$$

and we can define an automorphism α of $C[t, t^{-1}] \oplus C[t, t^{-1}]$ as the identity on P' and multiplication by t on P. With respect to the canonical basis of $C[t, t^{-1}] \oplus C[t, t^{-1}]$,

$$\alpha = \tfrac{1}{2}\begin{pmatrix} t(1+y) + 1 - y & -tx + x \\ -tx + x & t(1-y) + 1 + y \end{pmatrix}.$$

The matrix α has determinant equal to t and thus lies in $\mathrm{GL}_2(\mathrm{C}[\mathrm{t}, \mathrm{t}^{-1}])$. According to Theorem 7.4 of [1] its class in $K_1(C[t, t^{-1}])$ is the image of P by the canonical injection λ mentioned in §2. It is easy to see that P is not free over C. In fact it turns out to represent the non trivial class of $\mathrm{Pic}(C) = \mathbf{Z}/2$. Since the homomorphism ι in the cartesian square that defines A is surjective, tensoring the diagram with $\mathbf{R}[t, t^{-1}]$ yields a Milnor patching diagram

$$\begin{array}{ccc} A[t, t^{-1}] & \longrightarrow & \mathbf{R}[X, Y][t, t^{-1}] \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{R}[t, t^{-1}] & \xrightarrow{\iota} & C[t, t^{-1}] \end{array}$$

We can use this diagram and the matrix α (see for instance [1], Chapter IX, Theorem 5.1) to patch a rank 2 free module Q over $\mathbf{R}[X, Y][t, t^{-1}]$ with a rank 2 free module R over $\mathbf{R}[t, t^{-1}]$ and get a rank 2 projective module

$$M = \{(q, r) \in Q \times R \mid \alpha(\pi_*(q)) = \iota_*(r)\}$$

over $A[t, t^{-1}]$. We now equip M with a skew-symmetric structure. To do this we put on Q and on R the skew-symmetric structures defined, respectively, by the matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1/t \\ -1/t & 0 \end{pmatrix}.$$

Since $\alpha^* \tau \alpha = \sigma$, the skew-symmetric structures $\sigma \colon Q \to Q^*$ and $\tau \colon R \to R^*$ are compatible with the patching and therefore they define a skew-symmetric structure $\varphi \colon M \to M^*$ on M.

We claim that the class of this space is not in the image of $W'([t, t^{-1}])$. Extending to K_{-1} the Mayer-Vietoris sequence associated to (1) (see [1], Chapter XII, Theorem 8.3) we get an exact sequence

$$K_0(\mathbf{R}[X, Y]) \oplus K_0(\mathbf{R}) \to K_0(C) \to K_{-1}(A) \to K_{-1}(\mathbf{R}[X, Y]) \oplus K_{-1}(\mathbf{R}).$$

From the fact that regular rings have a vanishing K_{-1}, that $K_0(\mathbf{R}[X,Y]) = K_0(\mathbf{R}) = \mathbf{Z}$ and that $K_0(C) = \mathbf{Z} \oplus \mathbf{Z}/2$, where the element of order 2 is the class of P, we easily deduce that $K_{-1}(A) = \mathbf{Z}/2$, generated by the image of M. Thus, by Corollary 2.4, the class of M generates $H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = \mathbf{Z}/2$. Consider now the homomorphism

$$\omega\colon W(A[t,t^{-1}]) \longrightarrow H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A))$$

obtained by associating to any space its underlying projective module. Since $\omega((M,\varphi)) \neq 0$, (M,φ) cannot be Witt equivalent to a space supported by a module extended from A. This shows that the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ is not surjective.

REMARK 8.3. We suspect that even if the assumption of (a) is satisfied the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ may not be injective, but we did not find an example to confirm our suspicion.

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REFERENCES

[1] BASS, H. *Algebraic K-Theory.* Benjamin, 1969.
[2] BASS, H., A. HELLER and R. G. SWAN. The Whitehead group of a polynomial extension. *Inst. Hautes Études Sci. Publ. Math. 22* (1964), 61–79.
[3] KAROUBI, M. Localisation de formes quadratiques, II. *Ann. Sci. École Norm. Sup. (4) 8* (1975), 99–155.
[4] KNUS, M.-A. *Quadratic and Hermitian Forms over Rings.* Grundlehren der math. Wiss. 294. Springer, 1991.
[5] RANICKI, A. A. Algebraic L-theory. *Comment. Math. Helv. 49* (1974), 137–167.

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