§3. Periodic orbits

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **04.06.2024**

COVERING LEMMA. *Take a compact manifold S and consider a regular (normal) covering $T \to S$ with the action of $\Gamma = \pi_1(S)/\pi_1(T)$. Fix a fundamental domain $D \subset T$ and denote by $N(U)$, $U \subset X$, the number of motions $\gamma \in \Gamma$ such that the intersection $\gamma(U) \cap D$ is not empty. Consider an action of the group $\mathbf{R}$ of reals in S and its lifting to T.*

The entropy h of the action of $\mathbf{R}$ in S satisfies

$$h \geq \liminf_{r \to \infty} \frac{1}{|r|} \log N(r(D)),$$

where $r(D)$ denotes the image of D under the lifted action of $r \in \mathbf{R}$ in T.

Proof. Use the definition of entropy involving coverings.

This lemma (and the proof) holds for discrete time systems and immediately implies Manning's estimate of the topological entropy of an $f: S \to S$ in terms of the spectral radius of $f_*: H_1(S; \mathbf{R}) \to H_1(S; \mathbf{R})$. See [Ma], [Pu]. In Appendix 5 we show how to make use of the whole group $\pi_1(S)$.

§ 3. PERIODIC ORBITS

For maps $f: S \to S$ there are several ways to estimate from below the number $\operatorname{card}(\operatorname{Fix}(f^m))$ of all points of period m. Denote by $L(f)$ the Lefschetz number $\sum_{i=0}^{i=n}(-1)^i \operatorname{Trace}(f_{*i})$, where $i = \dim S$ and $f_{*i}: H_i(S; \mathbf{R}) \to H_i(S; \mathbf{R})$.

(***L***) *If all periodic points are nondegenerate (say, f is smooth and generic), then* $\operatorname{card}(\operatorname{Fix}(f^m)) \geq |L(f)|$ (Lefschetz).

(***Sh-S***) *If f is smooth and $\lim_{m \to \infty} |L(f^m)| = \infty$, then*

$$\lim_{m \to \infty} \operatorname{card}(\operatorname{Fix}(f^m)) = \infty$$

(Shub and Sullivan, see [Sh-S]).

(***Nie***) *Generally there is no way to extend the (**L**)-estimate to all maps, but in the presence of the fundamental group one can apply the Nielsen theory of fixed-point classes* (see [Nie] and Appendix 6). *This theory yields in many cases the estimate*

$$\operatorname{card}(\operatorname{Fix}(F)) \geq const\, |L(f)|,$$

and sometimes even $\operatorname{card}(\operatorname{Fix}(f^m)) \geq |L(F^m)|$, *where f is an arbitrary continuous map.*

EXAMPLE. Let S be a compact cell complex with homotopy type of a torus and let $f\colon S \to S$ be a continuous map such that $f_{*1}\colon H_1(S;\mathbf{R}) \to H_1(S;\mathbf{R})$ is hyperbolic (no eigenvalues with norm 1). Then

$$\operatorname{card}\big(\operatorname{Fix}(f^m)\big) \geq |L(f^m)| \geq C^m - 1$$

for some $C > 1$, and the closure of the union $\bigcup_{m=1}^{\infty} \operatorname{Fix}(f^m)$ contains a Cantor set.

REMARK. This example allows one to detect periodic points in Smale's horseshoe by homological means. A horseshoe is a space X with three subspaces A, B, Z and a map $f\colon X \to X$ with the properties:

(a) f sends $A \cup B$ into A and Z into B;

(b) Z separates A from B, i.e. there exists a function $a\colon X \to \mathbf{R}$ which is positive on A, negative on B and with $a^{-1}(0) \subset Z$.

(***Sm***) *If X, A and B are closed balls, then* $\operatorname{card}\big(\operatorname{Fix}(f^m)\big) < \frac{2^m - 1}{2}$ (Smale).

Proof. Take another copy X' of X and identify each point $x \in A \cup B$ with the corresponding point $x' \in A' \cup B' \subset X'$. Denote by Y the factor of $X \cup X'$ with that identification and construct a map $g\colon Y \to Y$ as follows:

- if $y \in X \subset Y$ and $a(y) \geq 0$, then $g(y) = f(y)$;
- if $y \in X$ and $a(y) < 0$, then $g(y) = (f(y))'$, where $()'$ means the involution permuting X and X' in Y;
- if $y \in X'$, then $g(y) = (g(y'))'$.

Since Y has the homotopy type of the circle, $|L(g^m)| = 2^m - 1$, and thus $\operatorname{card}\big(\operatorname{Fix}(g^m)\big) \geq 2^m - 1$. Projecting $Y \to X$ represents f as a factor of g that gives Smale's estimate.

This representation of f explains also (via Manning's estimate) why horseshoes have positive entropy.

CLOSED GEODESICS

Dealing with closed geodesics in a closed Riemannian manifold V we replace the Lefschetz numbers by the Betti numbers b_i of the space of maps from the circle S^1 to V. We set $M_m = M_m(V) = \frac{1}{m}\sum_{i=0}^{m} b_i$. The Morse theory provides the following estimate for the number $N_m = N_m(V)$ of simple closed geodesics of length $\leq m$:

(***Mo***) *If V is simply connected and all closed geodesics are nondegenerate (generic case), then $N_m \geq C M_m - 1$ for some $C > 0$* (see [Gr]).

(Probably, for most manifolds, M_m grows exponentially.)

In the degenerate case the situation is much more difficult, but still:

(***G-M***) $\limsup_{i\to\infty} b_i = \infty$ *implies* $\lim_{m\to\infty} N_m = \infty$ (Gromoll and Meyer, see [G-M]).

(About recent progress, see Klingenberg's lectures [Kl].)

The Nielsen theory collapses to a triviality in the geodesic case:

In each class of free homotopy of maps $S^1 \to V$ there is a closed geodesic; if it represents an indivisible element in $\pi_1(V)$, then every closed geodesic from that class is simple.

The estimate for N_m contained in this statement is exact for manifolds of negative curvature. For such manifolds $N_m \geq C^m - 2$ for some $C > 1$ (Sinai, see Appendix 7). But even for manifolds homeomorphic to the 2-torus it is unknown whether the estimate $N_m \geq C m^2 - 1$ is the best possible.

We give now three examples having no (?) discrete time analogs and demonstrating further connections between fundamental groups and closed geodesics. Proofs are more or less obvious and so omitted.

1. Suppose that the group $\pi_1(V)$ contains a (noncommutative) nilpotent subgroup Γ without torsion. Take a $\gamma \in [\Gamma, \Gamma]$, where $[\Gamma, \Gamma]$ denotes the commutator subgroup of Γ and γ is not the identity element, and denote by Z the (free cyclic) group generated by γ. Denote by N_m^Z the number of closed geodesics of length $\leq m$ representing elements from Z. Then $N_m^Z \geq C m - 1$ and there are infinitely many divisible elements in Z represented by simple closed geodesics; these geodesics can be chosen shortest, each in its homotopy class.

2. There is a non-empty class B of finitely presented groups such that if $\pi_1(V) \in B$, then there exists an infinite sequence g_i of simple closed contractible geodesics in V such that each g_i provides local minimum to the length functional and length$(g_i) \to \infty$ as $i \to \infty$. For example, B contains all groups with unsolvable word problem. (See Appendix 8 for further information.)

3. In order to make use of π_1 in locating other (not locally minimal) closed geodesics without non-degeneracy condition, one has to extend [Gr] to the non simply connected situation. When V is homeomorphic to $V_0 \times S^1$ and V_0 is simply connected, we can apply [Gr] directly and get $N_m(V) \geq C \log(m)$ for some $C > 0$. (Probably this is true when $H_1(V)$ is infinite or at least when $\pi_1(V) = \mathbf{Z}$.) The last estimate can be sharpened and we show this here for the simplest example when V_0 is the sphere S^3 and the proof is obvious [Gr].

Let V be homeomorphic to $S^3 \times S^1$. Then there exist closed geodesics $g_j^i \subset V$ (not necessarily simple) such that

1. *Each g_j^i, $i,j = 1,2,\ldots$, represents $\gamma^i \in \pi_1(V)$ where γ is a generator in $\pi_1(V)$.*
2. *For each i the geodesic g_1^i is the shortest in its homotopy class.*

Denote by $|g_j^i|$ the length of g_j^i.

3. $|g_1^{i+k}| + C \geq |g_1^i| + |g_1^k| \geq |g_1^{i+k}|$, *where $C \geq 0$ and $i,k = 1,2,\ldots$*
4. $|g_j^i| + C \geq |g_{j+1}^i| \geq |g_j^i|$ *for some $C > 0$ and $i,j = 1,2,\ldots$*
5. $\big||g_j^i| - |g_j^k|\big| \leq C|i-k|$ *for some $C > 0$ and $i,j,k = 1,2,\ldots$*
6. $|g_j^i| \geq \frac{i}{C}$ *for some $C > 0$ and $i,j = 1,2,\ldots$*

COROLLARY. *If V is as above, then* $\limsup_{m\to\infty} \frac{N_m(V)}{m^2} \geq$ *const* > 0.

All our estimates give a rather poor approximation to the (unknown) reality. Probably, in most cases N_m grows exponentially. That is so, of course, for "C^0-generic" manifolds ("C^0-generic" is used for C^0-generic manifolds having uncountably many closed geodesics).

REFERENCES

[Fr] FRANKS, J. Anosov diffeomorphisms. In: *Global Analysis, Berkeley, 1968.* Proc. Symp. Pure Math. 14, Amer. Math. Soc. 1970, 61–93.

[G-M] GROMOLL, D. and W. MEYER. Periodic geodesics on compact manifolds. *J. Differential Geom. 3* (1969), 493–510.

[Gr] GROMOV, M. Homotopical effects of dilatation. *J. Differential Geom. 13* (1978), 303–310.

[Kl] KLINGENBERG, W. *Lectures on Closed Geodesics.* Third ed. Mathematisches Institut der Universität Bonn, 1977.