3. In order to make use of π_1 in locating other (not locally minimal) closed geodesics without non-degeneracy condition, one has to extend [Gr] to the non simply connected situation. When V is homeomorphic to $V_0 \times S^1$ and V_0 is simply connected, we can apply [Gr] directly and get $N_m(V) \geq C \log(m)$ for some $C > 0$. (Probably this is true when $H_1(V)$ is infinite or at least when $\pi_1(V) = \mathbf{Z}$.) The last estimate can be sharpened and we show this here for the simplest example when V_0 is the sphere S^3 and the proof is obvious [Gr].

Let V be homeomorphic to $S^3 \times S^1$. Then there exist closed geodesics $g_j^i \subset V$ (not necessarily simple) such that

1. *Each g_j^i, $i,j = 1,2,\dots$, represents $\gamma^i \in \pi_1(V)$ where γ is a generator in $\pi_1(V)$.*
2. *For each i the geodesic g_1^i is the shortest in its homotopy class.*

Denote by $|g_j^i|$ the length of g_j^i.

3. $|g_1^{i+k}| + C \geq |g_1^i| + |g_1^k| \geq |g_1^{i+k}|$, *where $C \geq 0$ and $i,k = 1,2,\dots$*
4. $|g_j^i| + C \geq |g_{j+1}^i| \geq |g_j^i|$ *for some $C > 0$ and $i,j = 1,2,\dots$*
5. $\left| |g_j^i| - |g_j^k| \right| \leq C|i-k|$ *for some $C > 0$ and $i,j,k = 1,2,\dots$*
6. $|g_j^i| \geq \frac{i}{C}$ *for some $C > 0$ and $i,j = 1,2,\dots$*

COROLLARY. *If V is as above, then* $\limsup_{m\to\infty} \frac{N_m(V)}{m^2} \geq$ *const* > 0.

All our estimates give a rather poor approximation to the (unknown) reality. Probably, in most cases N_m grows exponentially. That is so, of course, for "C^0-generic" manifolds ("C^0-generic" is used for C^0-generic manifolds having uncountably many closed geodesics).

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WHY THE APPENDICES WERE NOT WRITTEN: AUTHOR'S APOLOGIES TO THE READERS

APPENDIX 2. The stable homeomorphism suggests a geometric link between the homotopy and topological invariance of Pontryagin classes, at least for manifolds with negative curvature but I did not manage to forge this to my satisfaction till 1996 (see [Gro_2]); also see [Fa-Jo] for a deeper analysis.

APPENDIX 3. One can define a notion of hyperbolicity for an automorphism α of an arbitrary finitely generated group Γ, such that (Γ, α) functorially defines a Bowen-Franks hyperbolic system (see [Gro_1]). Unfortunately, this class of (Γ, α) is rather limited, e.g. is not closed under free products and does not include hyperbolic automorphisms of surface groups. I still do not know what the right setting is.

APPENDIX 4. An obvious example of semi-hyperbolicity is provided by non-strictly expanding endomorphisms, where the geometric picture is rather clear. However, I still do not see a functorial description, in the spirit of the symbolic dynamics, of more general semi-hyperbolic systems, not even for the geodesic (or Weil chamber) flows on locally symmetric spaces (compare [B-G-S] and [Br-Ha]).

APPENDIX 5. The section on entropy was inspired by Manning's paper [Ma_1], but I was unaware of the prior paper by Dinaburg (see [Din]) that essentially contained the entropy estimate for geodesic flows (also discussed in [Ma_2]). On the other hand, estimating the entropy of an endomorphism (or an automorphism) f in terms of $f_*: \pi_1 \to \pi_1$ appears now much less clear than it seemed to me back in 1976. It is not hard to bound the entropy from below