1. Introduction

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GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS

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1. INTRODUCTION

For a C^*-algebra A, let $K_0(A)$, $K_1(A)$ be its K-theory groups. Thus $K_0(A)$ is the algebraic K_0-theory of the ring A and $K_1(A)$ is the algebraic K_0-theory of the ring $A \otimes C_0(\mathbf{R}) = C_0(\mathbf{R}, A)$. If $A \to B$ is a morphism of C^*-algebras, then there are induced homomorphisms of abelian groups $K_i(A) \to K_i(B)$. Bott periodicity provides a six term K-theory exact sequence for each exact sequence $0 \to J \to A \to B \to 0$ of C^*-algebras.

Discrete groups, Lie groups, group actions and foliations give rise through their convolution algebra to a canonical C^*-algebra, and hence to K-theory groups. The analytical meaning of these K-theory groups is clear as a receptacle for indices of elliptic operators. However, these groups are difficult to compute. For instance, in the case of semi-simple Lie groups, the free abelian group with one generator for each irreducible discrete series representation is contained in $K_0\, C^*G$ where C^*G is the reduced C^*-algebra of G. Thus an explicit determination of the K-theory in this case in particular involves an enumeration of the discrete series.

In this note we shall introduce a geometrically defined K-theory which specializes to discrete groups, Lie groups, group actions, and foliations. Its main features are its computability and the simplicity of its definition. In the case of semi-simple Lie groups it elucidates the role of the homogeneous space G/K (K the maximal compact subgroup of G) in the Atiyah-Schmid geometric construction of the discrete series [4]. Using elliptic operators we construct a natural map from our geometrically defined K-theory groups to the

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above analytic (i.e. C^*-algebra) K-theory groups. In all computed examples this map is an isomorphism. The picture that emerges is of two parallel theories: one analytic and one geometric. Elliptic operators provide a map from the geometric to the analytic theory. We give evidence for the conjecture that this map is always an isomorphism. In particular we prove that the map is injective for foliations with negatively curved leaves. We then explore some corollaries of this isomorphism conjecture. The injectivity is related through the work of G. G. Kasparov and A. S. Miscenko to the Novikov higher signature problem. We show how this problem leads to a conjecture on the invariance of certain foliation characteristic classes under leaf-wise homotopy equivalence. The surjectivity is related to a number of well-known C^*-algebra problems, such as the non-existence of idempotents in the reduced C^*-algebra of any torsion-free discrete group.

2. Lie group actions

G denotes a Lie group and X denotes a C^∞-manifold without boundary. Both G and X are assumed to be Hausdorff and second countable. G and X may have countably many connected components. G may be a countable discrete group.

DEFINITION 1. A C^∞ (right) action $X \times G \to X$ of G on X is *proper* if the map $X \times G \to X \times X$ given by

$$(x, g) \mapsto (x, xg)$$

is proper (i.e. the inverse image of any compact set is compact).

TERMINOLOGY. A *G-manifold* is a C^∞-manifold with a given (right) C^∞ G-action. If X, Y are G-manifolds a *G-map* from X to Y is a C^∞ G-equivariant map $f: X \to Y$. A G-manifold X is *proper* if the action of G on X is proper. A subset Δ of a proper G-manifold is *G-compact* if the image of Δ in the quotient space X/G is compact. A *G-vector bundle* on a G-manifold X is a C^∞-vector bundle E on X such that E is itself a G-manifold, the projection $E \to X$ is a G-map, and for each $(x, g) \in X \times G$ the map $E_x \to E_{xg}$ given by

$$u \mapsto ug \qquad (u \in E_x)$$

is linear.