2. Theorems on plane polygons

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **05.06.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

2. THEOREMS ON PLANE POLYGONS

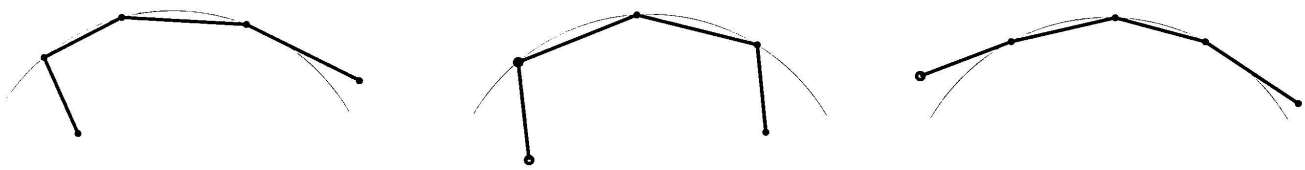
In this section we formulate our results for plane polygonal curves. The proofs will be given in Section 4.1.

2.1 DISCRETE 4-VERTEX THEOREM

The *osculating circle* of a smooth plane curve at a point is the circle (or straight line) that has 3rd order of contact with the curve at the given point. One may say that the osculating circle goes through 3 infinitely close points; at a vertex the osculating circle passes through 4 infinitely close points. Moreover, a generic curve crosses the osculating circle at a generic point and stays on one side of it at a vertex. This well-known fact motivates the following definition.

Let P be a plane convex n-gon; throughout this section we assume that $n \geq 4$. Denote the consecutive vertices by $V_1, \dots, V_n$; the subscripts are understood cyclically, that is, $V_{n+1} = V_1$, etc.

DEFINITION 2.1. A triple of vertices (V_i, V_{i+1}, V_{i+2}) is said to be *extremal* [1]) if V_{i-1} and V_{i+3} lie on the same side of the circle through V_i, V_{i+1}, V_{i+2} (this does not exclude the case where V_{i-1} or V_{i+3} belongs to the circle).



a) not extremal b) extremal

FIGURE 1

The next result follows from a somewhat more general theorem due to O. Musin and V. Sedykh [12] (see also [13]).

1) We have a terminological difficulty here: as we are dealing with polygons, we cannot use the term "vertex" in the same sense as in the smooth case; hence the term "extremal".

THEOREM 2.2. *Every plane convex polygon P has at least 4 extremal triples of vertices.*

EXAMPLE 2.3. If P is a quadrilateral then the theorem holds tautologically since the $(i-1)^{\mathrm{st}}$ vertex coincides with the $(i+3)^{\mathrm{rd}}$ for every i.

REMARK 2.4. An alternative approach to discretization of the 4-vertex theorem consists in inscribing circles in consecutive triples of sides of a polygon (the centre of such a circle is the intersection point of the bisectors of consecutive angles of the polygon). Then a triple of sides $(\ell_i, \ell_{i+1}, \ell_{i+2})$ is said to be *extremal* if the lines ℓ_{i-1}, ℓ_{i+3} either both intersect the corresponding circle or both fail to intersect it. With this definition an analogue of Theorem 2.2 holds true [19, 16], and this, in the limit, also provides the smooth 4-vertex theorem.

Both formulations, concerning circumscribed or inscribed circles, make sense on the sphere. Moreover, they are equivalent via projective duality.

2.2 DISCRETE THEOREM ON 6 AFFINE VERTICES

Five generic points in the plane determine a conic. Considering the plane as an affine part of the projective plane, the complement of the conic has two connected components. Let P be a plane convex n-gon; throughout this section we assume that $n \geq 6$. As in the previous section, we introduce the following definition.

DEFINITION 2.5. Five consecutive vertices $V_i, \dots, V_{i+4}$ are said to be *extremal* if V_{i-1} and V_{i+5} lie on the same side of the conic through these 5 points (this does not exclude the case where V_{i-1} or V_{i+5} belongs to the conic).

If P is replaced by a smooth convex curve, and $V_i, \dots, V_{i+4}$ are infinitely close points, we recover the definition of an affine vertex. Hence the following theorem is a discrete version of the smooth theorem on 6 affine vertices.

THEOREM 2.6. *Every plane convex polygon P has at least 6 extremal quintuples of vertices.*

EXAMPLE 2.7. If P is a hexagon then the theorem holds tautologically for the same reason as in Example 2.3.

REMARK 2.8. On interchanging sides and vertices, and replacing circumscribed conics by inscribed ones, we arrive at a "dual" theorem. The latter is equivalent to Theorem 2.6 via projective duality – cf. Remark 2.4.

2.3 DISCRETE GHYS THEOREM

A discrete object of study in this section is a pair of cyclically ordered n-tuples $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ in $\mathbf{RP}^1$ with $n \geq 4$. We choose an orientation of $\mathbf{RP}^1$ and assume that the cyclic ordering of each of the two n-tuples is induced by this orientation.

Recall that an ordered quadruple of distinct points in $\mathbf{RP}^1$ determines a number, the *cross-ratio*, which is a projective invariant. Choosing an affine parameter such that the points are given by real numbers $a < b < c < d$, the cross-ratio is

$$[a,b,c,d] = \frac{(c-a)(d-b)}{(b-a)(d-c)}. \tag{2.1}$$

DEFINITION 2.9. A triple of consecutive indices $(i, i+1, i+2)$ is said to be *extremal* if the difference of cross-ratios

$$[y_j, y_{j+1}, y_{j+2}, y_{j+3}] - [x_j, x_{j+1}, x_{j+2}, x_{j+3}] \tag{2.2}$$

changes sign as j varies from $i-1$ to i (this does not exclude the case where either of the differences vanishes).

THEOREM 2.10. *For every pair X, Y of n-tuples of points as above, there exist at least four extremal triples.*

EXAMPLE 2.11. If $n = 4$ then the theorem holds for a very simple reason. A cyclic permutation of four points induces the following transformation of their cross-ratio:

$$[x_4, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3, x_4]}{[x_1, x_2, x_3, x_4] - 1}, \tag{2.3}$$

and this is an involution. Furthermore, if $a > b > 1$ then $a/(a-1) < b/(b-1)$. Therefore, each triple of indices is extremal.

Let us interpret Theorem 2.10 in geometrical terms like Theorems 2.2 and 2.6. There exists a unique projective transformation that carries x_i, x_{i+1}, x_{i+2} into y_i, y_{i+1}, y_{i+2}, respectively. The graph G of this transformation can be seen as a curve in $\mathbf{RP}^1 \times \mathbf{RP}^1$; the three points (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}) lie on this graph. An ordered pair of points (x_j, x_{j+1}) in oriented $\mathbf{RP}^1$ defines a unique segment. An ordered pair of points $\big((x_j, y_j), (x_{j+1}, y_{j+1})\big)$ in $\mathbf{RP}^1 \times \mathbf{RP}^1$ also defines a unique segment, namely the one whose projection on each factor is a segment in $\mathbf{RP}^1$ as defined before. The triple $(i, i+1, i+2)$ is extremal if and only if the topological intersection index of the broken line $(x_{i-1}, y_{i-1}), \ldots, (x_{i+3}, y_{i+3})$ with the graph G is zero. This fact can be checked from (2.1) by a direct computation, which we omit.

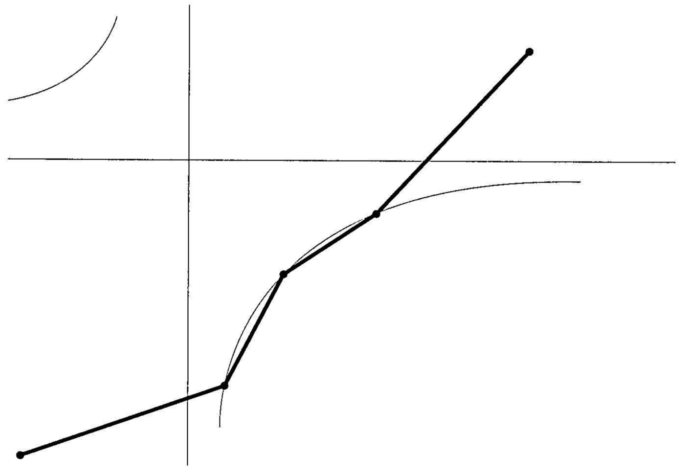


FIGURE 2

Let us also comment on the relation between Definition 2.9 and the zeroes of the Schwarzian derivative of a diffeomorphism of the projective line. Let

$$x_0 = 0, \quad x_1 = \varepsilon, \quad x_2 = 2\varepsilon, \quad x_3 = 3\varepsilon$$

be four infinitely close points given in some affine coordinate, and let $y_i = f(x_i)$ where f is a diffeomorphism of $\mathbf{RP}^1$. Then a direct computation using (2.1) yields:

$$[y_0, y_1, y_2, y_3] - [x_0, x_1, x_2, x_3] = \varepsilon^2\, S(f)(0) + O(\varepsilon^3),$$

where

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of f. Thus, for $\varepsilon \to 0$, Definition 2.9 corresponds to the vanishing of the Schwarzian derivative.