

## 2.1 DISCRETE 4-VERTEX THEOREM

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.05.2024**

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## 2. THEOREMS ON PLANE POLYGONS

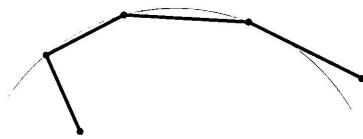
In this section we formulate our results for plane polygonal curves. The proofs will be given in Section 4.1.

### 2.1 DISCRETE 4-VERTEX THEOREM

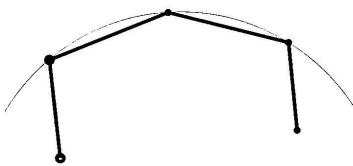
The *osculating circle* of a smooth plane curve at a point is the circle (or straight line) that has 3<sup>rd</sup> order of contact with the curve at the given point. One may say that the osculating circle goes through 3 infinitely close points; at a vertex the osculating circle passes through 4 infinitely close points. Moreover, a generic curve crosses the osculating circle at a generic point and stays on one side of it at a vertex. This well-known fact motivates the following definition.

Let  $P$  be a plane convex  $n$ -gon; throughout this section we assume that  $n \geq 4$ . Denote the consecutive vertices by  $V_1, \dots, V_n$ ; the subscripts are understood cyclically, that is,  $V_{n+1} = V_1$ , etc.

**DEFINITION 2.1.** A triple of vertices  $(V_i, V_{i+1}, V_{i+2})$  is said to be *extremal*<sup>1)</sup> if  $V_{i-1}$  and  $V_{i+3}$  lie on the same side of the circle through  $V_i, V_{i+1}, V_{i+2}$  (this does not exclude the case where  $V_{i-1}$  or  $V_{i+3}$  belongs to the circle).



a) not extremal



b) extremal

FIGURE 1

The next result follows from a somewhat more general theorem due to O. Musin and V. Sedykh [12] (see also [13]).

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<sup>1)</sup> We have a terminological difficulty here: as we are dealing with polygons, we cannot use the term “vertex” in the same sense as in the smooth case; hence the term “extremal”.

**THEOREM 2.2.** *Every plane convex polygon  $P$  has at least 4 extremal triples of vertices.*

**EXAMPLE 2.3.** If  $P$  is a quadrilateral then the theorem holds tautologically since the  $(i - 1)^{\text{st}}$  vertex coincides with the  $(i + 3)^{\text{rd}}$  for every  $i$ .

**REMARK 2.4.** An alternative approach to discretization of the 4-vertex theorem consists in inscribing circles in consecutive triples of sides of a polygon (the centre of such a circle is the intersection point of the bisectors of consecutive angles of the polygon). Then a triple of sides  $(\ell_i, \ell_{i+1}, \ell_{i+2})$  is said to be *extremal* if the lines  $\ell_{i-1}, \ell_{i+3}$  either both intersect the corresponding circle or both fail to intersect it. With this definition an analogue of Theorem 2.2 holds true [19, 16], and this, in the limit, also provides the smooth 4-vertex theorem.

Both formulations, concerning circumscribed or inscribed circles, make sense on the sphere. Moreover, they are equivalent via projective duality.

## 2.2 DISCRETE THEOREM ON 6 AFFINE VERTICES

Five generic points in the plane determine a conic. Considering the plane as an affine part of the projective plane, the complement of the conic has two connected components. Let  $P$  be a plane convex  $n$ -gon; throughout this section we assume that  $n \geq 6$ . As in the previous section, we introduce the following definition.

**DEFINITION 2.5.** Five consecutive vertices  $V_i, \dots, V_{i+4}$  are said to be *extremal* if  $V_{i-1}$  and  $V_{i+5}$  lie on the same side of the conic through these 5 points (this does not exclude the case where  $V_{i-1}$  or  $V_{i+5}$  belongs to the conic).

If  $P$  is replaced by a smooth convex curve, and  $V_i, \dots, V_{i+4}$  are infinitely close points, we recover the definition of an affine vertex. Hence the following theorem is a discrete version of the smooth theorem on 6 affine vertices.

**THEOREM 2.6.** *Every plane convex polygon  $P$  has at least 6 extremal quintuples of vertices.*

**EXAMPLE 2.7.** If  $P$  is a hexagon then the theorem holds tautologically for the same reason as in Example 2.3.