3. Main Theorem

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3. Main Theorem

All theorems from Section 2 are consequences of one theorem on the least number of flattenings of a closed polygon in real projective space.

In his remarkable work [3], M. Barner introduced the notion of a *strictly convex* curve in real projective space: this is a smooth closed curve $\gamma \subset \mathbf{RP}^d$ such that for every $(d-1)$-tuple of points on γ there exists a hyperplane through these points that does not intersect γ at any other points. Barner discovered the following theorem:

A strictly convex curve has at least $d+1$ distinct flattening points.

Recall that a flattening point of a projective space curve is a point at which the osculating hyperplane is stationary; in other words, this is a singularity of the projectively dual curve. In fact, Barner's result is considerably stronger but we shall not dwell on it here – see [15] for an exposition.

Our goal in this section is to provide a discrete version of Barner's theorem. First we need to develop an elementary intersection formalism for polygonal lines.

3.1 Intersection multiplicities

Throughout this section we shall look at closed polygons $P \subset \mathbf{RP}^d$ with vertices $V_1, \dots, V_n$ $(n \geq d+1)$ in general position. In other words, for every set of vertices $V_{i_1}, \dots, V_{i_k}$, where $k \leq d+1$, the span of $V_{i_1}, \dots, V_{i_k}$ is $(k-1)$-dimensional.

Definition 3.1. A polygon P is said to be *transverse* to a hyperplane H at a point $X \in P \cap H$ if

(a) X is an interior point of an edge and this edge is transverse to H, or

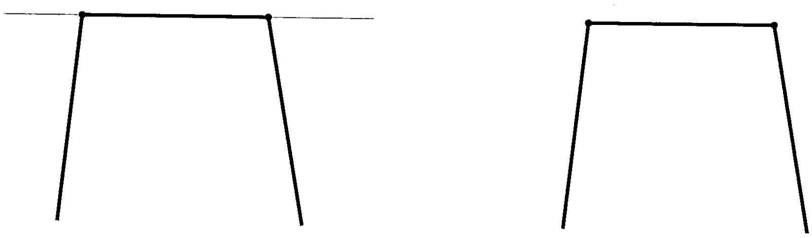
(b) X is a vertex, the two edges incident to X are transverse to H and are locally separated by H.

Clearly, transversality is an open condition.

Definition 3.2. A polygon P is said to intersect a hyperplane H with multiplicity k if for every hyperplane H' sufficiently close to H and transverse to P, the number of points $P \cap H'$ does not exceed k and, moreover, k is attained for some H'.

This definition does not exclude the case where a number of vertices of P lie in H.

H



H'

multiplicity 2

FIGURE 3

LEMMA 3.3. *Let $V_{i_1}, \ldots, V_{i_k}$ with $k \le d$ be vertices of P. Then any hyperplane H passing through $V_{i_1}, \ldots, V_{i_k}$ meets P with multiplicity at least k.*

Proof. Move each V_{i_j} ($j = 1, \ldots, k$) slightly along the edge (V_{i_j}, V_{i_j+1}) to obtain a new point V'_{i_j}. Let us show that a generic hyperplane H' through $V'_{i_1}, \ldots, V'_{i_k}$ is transverse to P. This will imply the lemma because H' has at least k intersections with P.

It suffices to show that H' does not contain any vertex of P. First we note that, since P is in general position, a generic hyperplane H through $V_{i_1}, \ldots, V_{i_k}$ does not contain any other vertex. The same holds true for every hyperplane which is sufficiently close to H. It remains to show that the chosen H' does not contain any of $V_{i_1}, \ldots, V_{i_k}$.

Suppose H' contains V_{i_j}. Then H' contains the edge (V_{i_j}, V_{i_j+1}) and therefore also V_{i_j+1}. If $i_j + 1 \notin \{i_1, \ldots, i_k\}$ we obtain a contradiction with the previous paragraph. If, on the other hand, $i_j + 1 \in \{i_1, \ldots, i_k\}$ then we can proceed in the same way with V_{i_j+1}. However, we cannot go on indefinitely since $k < n$. $\square$

The next definition is topological in nature.

DEFINITION 3.4. Consider a continuous curve in $\mathbf{RP}^d$ with endpoints A and Z. Let H be a hyperplane not containing A or Z. We say that A and Z *are on one side of* H if one can connect A and Z by a curve not intersecting H in such a way that the resulting closed curve is contractible. Otherwise we say that A and Z are *separated by* H.

Clearly, if one has only two points A and Z (and no curve connecting them), then one cannot say that the points are on one side of, or separated by, a hyperplane.

LEMMA 3.5. *Let $\Gamma = (A,\dots,Z)$ be a broken line in general position in $\mathbf{RP}^d$, and let H be a hyperplane not containing A or Z. Denote by k the intersection multiplicity of Γ with H. Then A and Z are separated by H if k is odd and not separated otherwise.*

Proof. Connect Z and A by a segment so as to obtain a closed polygon $\overline{\Gamma}$ and consider a hyperplane H' close to H, transverse to $\overline{\Gamma}$ and intersecting Γ in k points. Since $\overline{\Gamma}$ is contractible, H' intersects $\overline{\Gamma}$ in an even number of points. Therefore, H' intersects the segment (Z,A) for odd k and does not intersect it for even k. $\square$

The next definition introduces a significant class of polygons which is our main object of study.

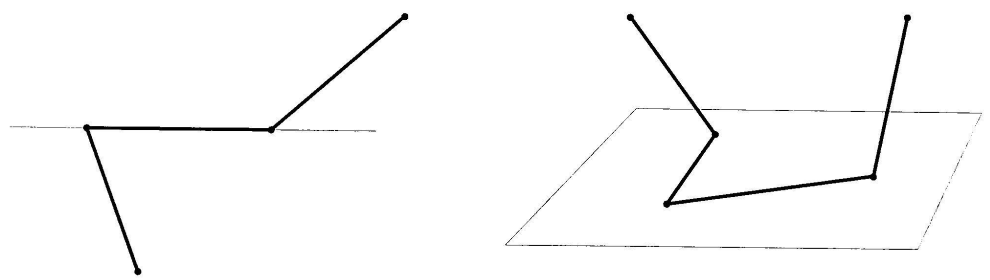
DEFINITION 3.6. A polygon P is called *strictly convex* if through every $d-1$ vertices there passes a hyperplane H whose intersection multiplicity with P is equal to $d-1$.

This definition becomes, in the smooth limit, that of strict convexity for smooth curves, due to Barner.

DEFINITION 3.7. A d-tuple of consecutive vertices $(V_i,\dots,V_{i+d-1})$ of a polygon P in $\mathbf{RP}^d$ is called a *flattening* if the endpoints V_{i-1} and V_{i+d} of the broken line $(V_{i-1},\dots,V_{i+d})$ are:

(a) separated by the hyperplane through $(V_i,\dots,V_{i+d-1})$ if d is even,

(b) not separated if d is odd.



a) $d=2$ b) $d=3$

FIGURE 4

REMARK 3.8. A curve in $\mathbf{RP}^d$ can be lifted to $\mathbf{R}^{d+1} \setminus \{0\}$; the lifting is not unique. Given a polygon $P \subset \mathbf{RP}^d$ with vertices $V_1, \dots, V_n$, we lift it to $\mathbf{R}^{d+1}$ as a polygon $\widetilde{P}$ and denote its vertices by $\widetilde{V}_1, \dots \widetilde{V}_n$. Then a d-tuple $(V_i, \dots, V_{i+d-1})$ is a flattening if and only if the determinant

$$\Delta_j = \left|\widetilde{V}_j \dots \widetilde{V}_{j+d}\right| \tag{3.1}$$

changes sign as j varies from $i-1$ to i.

This property is independent of the lifting.

3.2 A SIMPLEX IS STRICTLY CONVEX

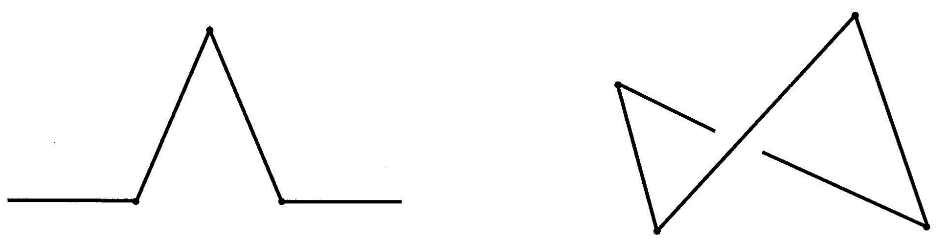
Define a simplex $S_d \subset \mathbf{RP}^d$ with vertices $V_1, \dots, V_{d+1}$ as the projection from the punctured $\mathbf{R}^{d+1}$ of the polygonal line:

$$\widetilde{V}_1 = (1,0,\dots,0), \quad \widetilde{V}_2 = (0,1,0,\dots,0), \quad \dots, \widetilde{V}_{d+1} = (0,\dots,0,1) \tag{3.2}$$

and

$$\widetilde{V}_{d+2} = (-1)^{d+1}\widetilde{V}_1 . \tag{3.3}$$

The last vertex has the same projection as the first one; S_d is contractible for odd d, and non-contractible for even d.



a) $d=2$ b) $d=3$

FIGURE 5

PROPOSITION 3.9. *The polygon S_d is strictly convex.*

Proof. We need to prove that through every $(d-1)$-tuple

$$(V_1, \dots, \widehat{V}_i, \dots, \widehat{V}_j, \dots, V_{d+1})$$

there passes a hyperplane H intersecting P with multiplicity $d-1$. Select a point W on the line $(\widetilde{V}_i, \widetilde{V}_j)$ in such a manner that W lies on the segment $(\widetilde{V}_i, \widetilde{V}_j)$ if $j-i$ is even, and does not lie on it if $j-i$ is odd. Define $\widetilde{H}$ as the linear span of $\widetilde{V}_1, \dots, \widehat{\widetilde{V}}_i, \dots, \widehat{\widetilde{V}}_j, \dots, \widetilde{V}_{d+1}, W$. We claim that its projection $H \subset \mathbf{RP}^d$ meets S_d with multiplicity $\leq d-1$.

Let H' be a hyperplane close to H and transverse to S_d; assume, further, that H' contains no vertices. It is enough to show that H' cannot intersect S_d in more than $d-1$ points. On the one hand, H' cannot intersect all the edges of S_d. Or else, $\widetilde{H'}$ would separate all pairs of consecutive vertices, and this would contradict the choice of W. On the other hand, if the number of intersections of H' and S_d were greater than $d-1$, it would be equal to $d+1$. Indeed, for topological reasons, the parity of this intersection number is that of $d-1$. We obtain a contradiction, which proves the claim.

Finally, by Lemma 3.3, the intersection multiplicity of H with S_d is not less than $d-1$. □

A curious property of a simplex is that each of its d-tuples of vertices is a flattening.

LEMMA 3.10. *The simplex S_d has $d+1$ flattenings.*

Proof. The determinant (3.1) involves all $d+1$ vectors $\widetilde{V}_1, \dots, \widetilde{V}_{d+1}$. If d is odd then, according to (3.3), $\widetilde{V}_{d+2} = \widetilde{V}_1$, and we are reduced to the fact that a cyclic permutation of vectors changes the sign of the determinant. On the other hand, if d is even then $\widetilde{V}_{d+2} = -\widetilde{V}_1$, which also leads to a change of sign in (3.1). □

3.3 BARNER'S THEOREM FOR POLYGONS

Now we formulate the result which serves as the main technical tool in the proof of Theorems 2.2, 2.6 and 2.10. Recall that we consider generic polygons in $\mathbf{RP}^d$ with at least $d+1$ vertices.

THEOREM 3.11. *A strictly convex polygon in $\mathbf{RP}^d$ has at least $d+1$ flattenings.*

Proof. Induction on the number n of vertices.

Induction starts with $n = d+1$. Up to projective transformations, the unique strictly convex $(d+1)$-gon is the simplex S_d. Indeed, every generic $(d+1)$-tuple of points in $\mathbf{RP}^d$ can be taken into any other one by a projective transformation. Therefore, all generic broken lines with d edges are projectively equivalent. It remains for us to connect the last point with the first one, and there are exactly two ways of doing this. One yields a contractible polygon, and the other a non-contractible one. One of these polygons is S_d, while the other one cannot be strictly convex, since its intersection number with a hyperplane does not have the same parity as $d-1$. The base for induction is then provided by Lemma 3.10.

Let P be a strictly convex $(n+1)$-gon with vertices $V_1, \dots, V_{n+1}$. Delete V_{n+1} and connect V_n with V_1 in such a way that the new edge (V_n, V_1), together with the two deleted ones, (V_n, V_{n+1}) and (V_{n+1}, V_1), form a contractible triangle. Denote the new polygon by P'.

Let us show that P' is strictly convex. P is strictly convex, therefore through every $d-1$ vertices of P' there passes a hyperplane H intersecting P with multiplicity $d-1$. We want to show that the intersection multiplicity of H with P' is also $d-1$. Let H' be a hyperplane close to H and transverse to P and P'. The intersection number of H' with P' does not exceed that with P. Indeed, if H' intersects the new edge, then it intersects one of the deleted ones since the triangle is contractible.

By the induction hypothesis, P' has at least $d+1$ flattenings. To prove the theorem, it remains for us to show that P' cannot have more flattenings than P.

Consider the sequence of determinants (3.1) $\Delta_1, \Delta_2, \dots, \Delta_{n+1}$. On replacing P by P' we remove $d+1$ consecutive determinants

$$\Delta_{n-d+1},\ \Delta_{n-d+2},\ \dots,\ \Delta_{n+1} \tag{3.4}$$

and replace them with d new determinants

$$\Delta'_{n-d+1},\ \Delta'_{n-d+2},\ \dots,\ \Delta'_{n}, \tag{3.5}$$

where

$$\Delta'_{n-d+i} = \left|\widetilde{V}_{n-d+i}\ \dots\ \widehat{\widetilde{V}}_{n+1}\ \dots\ \widetilde{V}_{n+i+1}\right| \tag{3.6}$$

with $i = 1, \dots, d$. The transition from (3.4) to (3.5) is done in two steps. Firstly, we add (3.5) to (3.4) so that the two sequences alternate, that is, we put Δ'_j between Δ_j and Δ_{j+1}. And secondly, we delete the "old" determinants (3.4). We will prove that the first step preserves the number of sign changes, while the second step obviously cannot increase this number.

LEMMA 3.12. *If Δ_{n-d+i} and $\Delta_{n-d+i+1}$ have the same sign, then Δ'_{n-d+i} is also of the same sign.*

Proof of the lemma. Since P is in general position, the removed vector $\widetilde{V}_{n+1}$ is a linear combination of $d+1$ vectors $\widetilde{V}_{n-d+i}, \dots, \widetilde{V}_n$, $\widetilde{V}_{n+2}, \dots, \widetilde{V}_{n+i+1}$:

$$\widetilde{V}_{n+1} = a\widetilde{V}_{n-d+i} + b\widetilde{V}_{n+i+1} + \cdots, \tag{3.7}$$

where the dots indicate a linear combination of the remaining vectors. It follows from (3.6) that

$$\Delta_{n-d+i} = (-1)^{i-1}\, b\, \Delta'_{n-d+i}, \quad \Delta_{n-d+i+1} = (-1)^{d-i}\, a\, \Delta'_{n-d+i}\,. \tag{3.8}$$

It is time to use the strict convexity of P. Let H be a hyperplane in $\mathbf{RP}^d$ through $d-1$ vertices $V_{n-d+i+1}, \ldots, \widehat{V}_{n+1}, \ldots, V_{n+i}$ which intersects P with multiplicity $d-1$, and let $\widetilde{H}$ be its lifting to $\mathbf{R}^{d+1}$. Choose a linear function φ in $\mathbf{R}^{d+1}$ vanishing on $\widetilde{H}$ and such that $\varphi(\widetilde{V}_{n+1}) > 0$. We claim that

$$(-1)^{d-i}\varphi(\widetilde{V}_{n-d+i}) > 0 \quad \text{and} \quad (-1)^{i-1}\varphi(\widetilde{V}_n) > 0\,. \tag{3.9}$$

Indeed, by Lemma 3.3, the intersection multiplicities of $\widetilde{H}$ with the polygonal lines $(\widetilde{V}_{n-d+i}, \ldots, \widetilde{V}_{n+1})$ and $(\widetilde{V}_{n+1}, \ldots, \widetilde{V}_{n+i+1})$ are at least $d-i$ and $i-1$, respectively. Since H intersects P with multiplicity $d-1$, the above two multiplicities are indeed equal to $d-i$ and $i-1$. The inequalities (3.9) now readily follow from Lemma 3.5.

Finally, we evaluate φ on (3.7):

$$\varphi(\widetilde{V}_{n+1}) = a\,\varphi(\widetilde{V}_{n-d+i}) + b\,\varphi(\widetilde{V}_{n+i+1})\,.$$

It follows from (3.9) and the inequality $\varphi(\widetilde{V}_{n+1}) > 0$ that at least one of the numbers $(-1)^{i-1}b$ and $(-1)^{d-i}a$ is positive. In view of (3.8), Lemma 3.12 follows. $\square$

Thus Theorem 3.11 is also proved. $\square$

REMARK 3.13. Strict convexity is necessary for the existence of $d+1$ flattenings. One can easily construct a closed polygon without any flattenings and even C^0-approximate an arbitrary closed smooth curve by such polygons. In the smooth case such an approximation is well known: given a curve γ_0, the approximating one, γ, spirals around in a tubular neighbourhood of γ_0. In the polygonal case we take a sufficiently fine straightening of γ.

4. APPLICATIONS OF THE MAIN THEOREM

4.1 PROOF OF THEOREMS 2.2, 2.6 AND 2.10

Now we prove the results announced in Section 2. The idea is the same in all three cases and is precisely that of Barner's proof of the smooth versions of these theorems – see [3] and also [15]. We will consider Theorem 2.6 in detail, indicating the necessary changes in the other two cases.