3.2 A SIMPLEX IS STRICTLY CONVEX

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REMARK 3.8. A curve in $\mathbf{RP}^d$ can be lifted to $\mathbf{R}^{d+1} \setminus \{0\}$; the lifting is not unique. Given a polygon $P \subset \mathbf{RP}^d$ with vertices $V_1, \ldots, V_n$, we lift it to $\mathbf{R}^{d+1}$ as a polygon $\widetilde{P}$ and denote its vertices by $\widetilde{V}_1, \ldots \widetilde{V}_n$. Then a d-tuple $(V_i, \ldots, V_{i+d-1})$ is a flattening if and only if the determinant

$$\Delta_j = \left|\widetilde{V}_j \ \ldots \ \widetilde{V}_{j+d}\right| \tag{3.1}$$

changes sign as j varies from $i-1$ to i.

This property is independent of the lifting.

3.2 A SIMPLEX IS STRICTLY CONVEX

Define a simplex $S_d \subset \mathbf{RP}^d$ with vertices $V_1, \ldots, V_{d+1}$ as the projection from the punctured $\mathbf{R}^{d+1}$ of the polygonal line:

$$\widetilde{V}_1 = (1, 0, \ldots, 0), \quad \widetilde{V}_2 = (0, 1, 0, \ldots, 0), \quad \ldots, \widetilde{V}_{d+1} = (0, \ldots, 0, 1) \tag{3.2}$$

and

$$\widetilde{V}_{d+2} = (-1)^{d+1} \widetilde{V}_1 \,. \tag{3.3}$$

The last vertex has the same projection as the first one; S_d is contractible for odd d, and non-contractible for even d.

a) $d=2$ b) $d=3$

FIGURE 5

PROPOSITION 3.9. *The polygon S_d is strictly convex.*

Proof. We need to prove that through every $(d-1)$-tuple

$$(V_1, \ldots, \widehat{V}_i, \ldots, \widehat{V}_j, \ldots, V_{d+1})$$

there passes a hyperplane H intersecting P with multiplicity $d-1$. Select a point W on the line $(\widetilde{V}_i, \widetilde{V}_j)$ in such a manner that W lies on the segment $(\widetilde{V}_i, \widetilde{V}_j)$ if $j-i$ is even, and does not lie on it if $j-i$ is odd. Define $\widetilde{H}$ as the linear span of $\widetilde{V}_1, \ldots, \widehat{\widetilde{V}}_i, \ldots, \widehat{\widetilde{V}}_j, \ldots, \widetilde{V}_{d+1}, W$. We claim that its projection $H \subset \mathbf{RP}^d$ meets S_d with multiplicity $\leq d-1$.

Let H' be a hyperplane close to H and transverse to S_d; assume, further, that H' contains no vertices. It is enough to show that H' cannot intersect S_d in more than $d-1$ points. On the one hand, H' cannot intersect all the edges of S_d. Or else, $\widetilde{H'}$ would separate all pairs of consecutive vertices, and this would contradict the choice of W. On the other hand, if the number of intersections of H' and S_d were greater than $d-1$, it would be equal to $d+1$. Indeed, for topological reasons, the parity of this intersection number is that of $d-1$. We obtain a contradiction, which proves the claim.

Finally, by Lemma 3.3, the intersection multiplicity of H with S_d is not less than $d-1$. □

A curious property of a simplex is that each of its d-tuples of vertices is a flattening.

LEMMA 3.10. *The simplex S_d has $d+1$ flattenings.*

Proof. The determinant (3.1) involves all $d+1$ vectors $\widetilde{V}_1, \ldots, \widetilde{V}_{d+1}$. If d is odd then, according to (3.3), $\widetilde{V}_{d+2} = \widetilde{V}_1$, and we are reduced to the fact that a cyclic permutation of vectors changes the sign of the determinant. On the other hand, if d is even then $\widetilde{V}_{d+2} = -\widetilde{V}_1$, which also leads to a change of sign in (3.1). □

3.3 BARNER'S THEOREM FOR POLYGONS

Now we formulate the result which serves as the main technical tool in the proof of Theorems 2.2, 2.6 and 2.10. Recall that we consider generic polygons in $\mathbf{RP}^d$ with at least $d+1$ vertices.

THEOREM 3.11. *A strictly convex polygon in $\mathbf{RP}^d$ has at least $d+1$ flattenings.*

Proof. Induction on the number n of vertices.

Induction starts with $n = d+1$. Up to projective transformations, the unique strictly convex $(d+1)$-gon is the simplex S_d. Indeed, every generic $(d+1)$-tuple of points in $\mathbf{RP}^d$ can be taken into any other one by a projective transformation. Therefore, all generic broken lines with d edges are projectively equivalent. It remains for us to connect the last point with the first one, and there are exactly two ways of doing this. One yields a contractible polygon, and the other a non-contractible one. One of these polygons is S_d, while the other one cannot be strictly convex, since its intersection number with a hyperplane does